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LECTURE 9

GRA 6035 MATHEMATICS

EIVIND ERIKSEN

OCT 20, 2016

Plan:

1. Envelope theorems
2. Bordered Hessians

Reading:

[ME] 19.2–19.3 (19.4–19.6)

Note: Plenary problem session Tue 25/10 at 17
Focus: Lecture 7-8

Review: Lagrange / Kuhn-Tucker problems

max/min $f(\underline{x})$ when $\begin{cases} g_1(\underline{x}) = a_1 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases}$

Lagrange pb.

max $f(\underline{x})$ when $\begin{cases} g_1(\underline{x}) \leq a_1 \\ \vdots \\ g_m(\underline{x}) \leq a_m \end{cases}$

Kuhn-Tucker pb.
(in std. form)

Method:

1. Solve FOC+C $\longrightarrow$ ordinary candidate pts
2. Determine if candidate pts are actually max/min
   - SOC: $\mathcal{L}(\underline{x}; \lambda_1^*, \ldots, \lambda_m^*)$ convex/concave $\Downarrow$ min / max
   - EVT: {adm pts} bounded $\Downarrow$ there is max/min

need to check NDCQ (special cand. pts).

adm. pts where NDCQ fails

FOC+C + CSC

$\lambda_i \geq 0$ and $\lambda_i \cdot (g_i(\underline{x}) - a_i) = 0$
$\Updownarrow$
$\lambda_i \geq 0$ if $g_i(\underline{x}) = a_i$ and
$\lambda_i = 0$ if $g_i(\underline{x}) < a_i$

for all i

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(i) Envelope theorems

"change in optimal value (max/min value) when we change the problem a little"

Unconstrained case:

max $f(x) = -x^2+2x+4$

$f'(x) = -2x+2 = 0$

$\underline{x=1}$

$f''(x) = -2$ negative

$H(f) = (f''(x)) = (-2)$ negative det.

$x=1$ is (local and global) max

max value: $f(1) = 5$

max $f(x;a) = -x^2+2ax+4$ (a parameter)

$f'(x) = -2x+2a = 0$

$\underline{x=a}$

$f''(x) = -2 < 0$ (f concave for all a)

$x^*(a) = a$ is (local and global) max

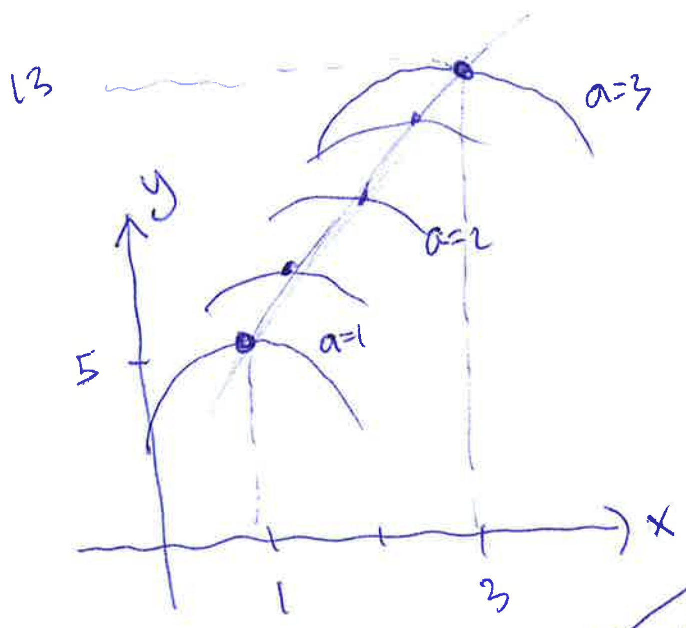
max. value:

$f^*(a) = f(x^*(a)) = -a^2+2a\cdot a+4 = \underline{a^2+4}$

↑ optimal value function

$$\frac{df^*(a)}{da} = 2a$$

envelope thm tells us how to compute $\frac{df^*(a)}{da}$



$f^*(2)=8$

$\underline{a=1}$: $x=1$, $f(1)=5$

$\underline{a=2}$: $f^*(2) \approx \underset{5}{f^*(1)} + (2-1)\cdot 2\cdot 1 = 5+2 = \underline{\underline{7}}$

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Envelope thm: Unconstrained case

If the unconstrained problem

$$\max/\min\ f(x;a) \qquad (a \text{ parameter})$$

has solution $x = x^*(a)$ and $f^*(a) = f(x^*(a))$, then

$$\frac{df^*(a)}{da} = \frac{\partial f}{\partial a}\bigg|_{x=x^*(a)} = \frac{\partial f}{\partial a}(x^*(a);a)$$

Ex: $\max\ f(x;a) = -x^2 + 2ax + 4$

$$\frac{\partial f}{\partial a} = 2x \qquad \frac{\partial f}{\partial a}\bigg|_{x=x^*(a)} = \underline{2\cdot x^*(a)} = \underline{2a}$$

$x^*(a) = a$

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Ex: ~~max~~/min $f(x,y;b) = x^2+3y^2-2x-by+7$

$$\frac{df^*(b)}{db} = \frac{\partial f}{\partial b}\Big|_{x=x^*(b),\, y=y^*(b)}$$

$$\uparrow$$

$$\frac{\partial f}{\partial b} = -y \implies \frac{\partial f}{\partial b}\Big|_{x=x^*(b),\, y=y^*(b)} = -y^*(b)$$

Compute: $(x^*(b), y^*(b))$

$$f'_x = 2x-2=0 \qquad x=1$$
$$f'_y = 6y-b=0 \qquad y=b/6$$

Stationary pts.

$$H(f) = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix} \qquad D_1=2,\ D_2=12$$

pos. defn.

f convex

$x=1,\ y=b/6$ is global min.

$x^*(b)=1,\ y^*(b)=\frac{b}{6}$

Alt I: Explicitly

$x^*(b)=1,\ y^*(b)=\frac{b}{6}$

$$f^*(b) = 1^2+3\cdot\left(\frac{b}{6}\right)^2-2\cdot 1 - b\cdot\frac{b}{6}+7$$

$$= 1+\frac{1}{12}b^2-2-b^2/6+7$$

$$= -\frac{1}{12}b^2+6$$

Alt 2: Envelope thm.

$$\frac{df^*(b)}{db} = -\frac{b}{6}$$

if $b>0$, then an increase in b will decrease the min. value

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Constrained case: Lagrange / Kuhn-Tucker pb.

Envelope thm: Lagrange case

If the Lagrange problem

$$\max/\min\ f(x_1,\dots,x_n;a) \quad \text{when} \quad \begin{cases} g_1(\underline{x};a)=0 \\ \vdots \\ g_m(\underline{x};a)=0 \end{cases}$$

(parameter a)

has an ordinary candidate point as max/min-pt

$$(x_1^*, x_2^*, \dots, x_n^*; \lambda_1^*, \dots, \lambda_m^*) = (x_1^*(a), x_2^*(a), \dots, x_n^*(a); \lambda_1^*(a), \dots, \lambda_m^*(a))$$

then

$$\frac{df^*(a)}{da} = \frac{\partial L}{\partial a}\Big|_{\underline{x}=\underline{x}^*(a);\ \underline{\lambda}=\underline{\lambda}^*(a)}$$

Exactly the same in the Kuhn-Tucker case.

Ex: max $x+3y$ when $x^2+y^2 \le 10$

a) Solve this problem

b) Use envelope thm. to approximate the max value of

$$\max\ x+4y \quad \text{when} \quad x^2+y^2 \le 10$$

Formulation: $\max f(x,y;a) = x+ay$ when $x^2+y^2-10 \le 0$

a) Solve for $a=3$ $\quad f^*(3)$

b) Solve for $a=4$ $\quad f^*(4)$

max $x+ay$ when $x^2+y^2-10 \le 0$

$$\boxed{\mathcal{L} = x + ay - \lambda(x^2+y^2-10)}$$

a) $\underline{a=3}$: $\mathcal{L} = x + 3y - \lambda \cdot (x^2+y^2-10)$

$$\mathcal{L}'_x = 1 - \lambda \cdot 2x = 0$$
$$\mathcal{L}'_y = 3 - \lambda \cdot 2y = 0$$

$x^2+y^2=10$	$x^2+y^2<10$
$\lambda \ge 0$	$\lambda = 0$
$x = \frac{1}{2\lambda}$	$1=0$ imp.
$y = 3/2\lambda$	$\Downarrow$ no sol'ns

$x^2+y^2=10$

$\frac{1}{4\lambda^2} + \frac{9}{4\lambda^2} = 10$

$4\lambda^2 = 1$

$\lambda^2 = 1/4$

$\lambda = 1/2$

$x=1$
$y=3$
$\lambda = 1/2$

$f = 10$

SOC: $\mathcal{L}(x,y;\frac{1}{2}) = x+3y - \frac{1}{2}(x^2+y^2-10)$

$\mathcal{L}'_x = 1-x$

$\mathcal{L}'_y = 3-y$

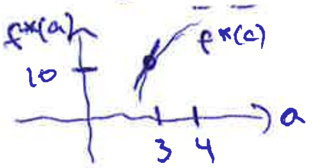
$$H(\mathcal{L}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad D_1 = -1, \quad D_2 = 1$$

neg. defn.
$\mathcal{L}(x,y;1/2)$ is concave

$$\left.\begin{aligned} x^*(3) &= 1 \\ y^*(3) &= 3 \\ \lambda^*(3) &= 1/2 \end{aligned}\right\} \quad \underline{f^*(3) = 10}$$

b) $\underline{\text{Envelope thm:}}$ $\frac{df^*(a)}{da} = \left.\frac{\partial \mathcal{L}}{\partial a}\right|_{x=x^*(a);\, \lambda=\lambda^*(a)} = y^*(a)$

$a=3$: $f^*(3)=10$
$y^*(3)=3$



$f^*(4) \simeq f^*(3) + 1 \cdot 3$
$= 10+3 = \underline{\underline{13}}$

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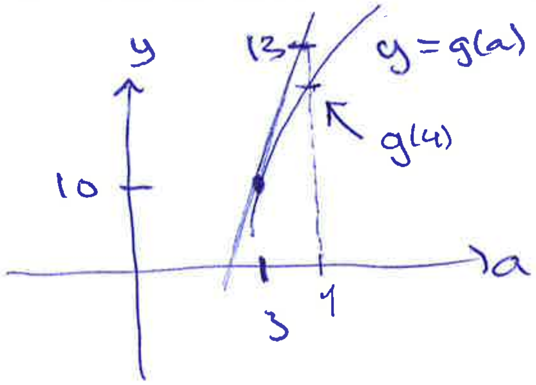

$g(a) = f^*(a)$

$g(3) = 10$

Env. thm. $\longrightarrow$ $g'(a) = y^*(a)$

$g'(3) = y^*(3) = 3$

$g(a) \cong g(3) + g'(3)\cdot(a-3)$

$y \cong y_0 + g'(a)\cdot(a-a_0)$

$g(a) \cong 10 + 3\cdot(a-3)$

$g(4) \cong 10 + 3\cdot(4-3)$

$= 10 + 3\cdot 1 = \underline{\underline{13}}$

Exact value of $f^*(4)$:

max $x+4y$ when $x^2+y^2 \le 10$

$\mathcal{L} = x + 4y - \lambda(x^2+y^2)$

$\mathcal{L}'_x = 1 - \lambda\cdot 2x = 0$

$\mathcal{L}'_y = 4 - \lambda\cdot 2y = 0$

$x^2+y^2=10$	$x^2+y^2<10$
$\lambda \ge 0$	$\lambda = 0$
$x = \frac{1}{2\lambda}$	$1 = 0$ imp.
$y = \frac{4}{2\lambda}$	no sol's

$x^2+y^2 = \frac{1}{4\lambda^2} + \frac{16}{4\lambda^2} = 10$

$4\lambda^2 = \frac{17}{10}$

$\lambda^2 = \frac{17}{40}$

$\lambda = \sqrt{\frac{17}{40}} = \frac{1}{2}\sqrt{\frac{17}{10}}$

$x = \sqrt{\frac{10}{17}}$

$y = 4\sqrt{\frac{10}{17}}$

$\lambda = \frac{1}{2}\sqrt{\frac{17}{10}}$

$f = \sqrt{\frac{10}{17}} + 4\sqrt{\frac{10}{17}}\cdot 4$

$= 17\cdot\sqrt{\frac{10}{17}} \cong \underline{\underline{13{,}038}}$

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max $x+3y$ when $x^2+y^2 \leq b$

$x^2+y^2-b \leq 0$

Change: $\underline{b=10}$ to $\underline{b=11}$

$x^*(10)=1$

$y^*(10)=3$

$f^*(10)=10$

$\lambda^*(10)=\underline{1/2}$

$$f^*(11) \approx f^*(10) + 1 \cdot \lambda^*(10)$$

$$= 10 + 1 \cdot \tfrac{1}{2} = \underline{\underline{10{,}5}}$$

$$L = x+3y - \lambda \cdot (x^2+y^2-b)$$

$$\frac{df^*(b)}{db} = \frac{\partial L}{\partial b}\Big|_{\underline{x}=\underline{x}^*(b),\ \underline{\lambda}=\lambda^*(b)}$$

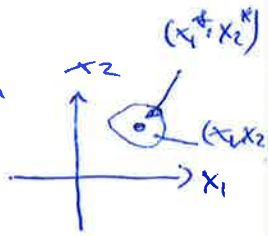
$$= \underline{\underline{\lambda^*(b)}}$$

<u>Interpretation of Lagrange multiplier λ:</u>

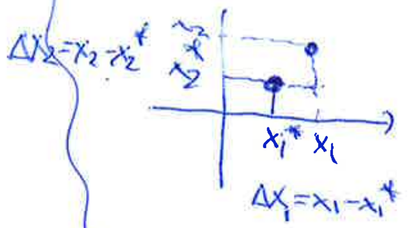
$$\lambda = \frac{df^*(a)}{da}$$ when a is the constant on the right-hand side of the <u>constraint</u>.

Linear approximation of a function

Let $f(x_1, \ldots, x_n)$ be a C^1 function in a neighbourhood at the point $(x_1^*, \ldots, x_n^*)$. Then the linear approximation of f is



$$\boxed{\begin{aligned} f(x_1, \ldots, x_n) \simeq f(x_1^*, \ldots, x_n^*) &+ \frac{\partial f}{\partial x_1}(x_1^*, \ldots, x_n^*) \cdot (x_1 - x_1^*) \\ &+ \frac{\partial f}{\partial x_2}(x_1^*, \ldots, x_n^*) \cdot (x_2 - x_2^*) \\ &+ \ldots + \frac{\partial f}{\partial x_n} \cdot (x_1^*, \ldots, x_n^*) \cdot (x_n - x_n^*) \end{aligned}}$$



Ex: $f(x,y) = e^{xy}$ at $(0,0)$ and $(1,1)$

$f(0,0) = \underline{1}$

$f'_x = e^{xy} \cdot y \quad f'_x(0,0) = \underline{0}$

$f'_y = e^{xy} \cdot x \quad f'_y(0,0) = \underline{0}$

When (x,y) close to $(0,0)$:

$$f(x,y) \simeq 1 + 0 \cdot (x-0) + 0 \cdot (y-0)$$
$$= 1$$

$f(1,1) = e \quad f'_x(1,1) = e \quad f'_y(1,1) = e$

When (x,y) close to $(1,1)$:

$$f(x,y) \simeq e + e \cdot (x-1) + e \cdot (y-1)$$
$$= ex + ey - e$$

2 Bordered Hessians.

Ex: max/min $f(x,y,z) = x^2y^2z^2$ wh $x^2+y^2+z^2=3$

$$\mathcal{L} = x^2 \cdot y^2 \cdot z^2 - \lambda \cdot (x^2+y^2+z^2)$$

$$\left.\begin{aligned}
\mathcal{L}'_x &= 2x\,y^2z^2 - \lambda \cdot 2x = 0 \\
\mathcal{L}'_y &= x^2 \cdot 2y \cdot z^2 - \lambda \cdot 2y = 0 \\
\mathcal{L}'_z &= x^2y^2 \cdot 2z - \lambda \cdot 2z = 0 \\
x^2+y^2+z^2 &= 3
\end{aligned}\right\}$$

$(x,y,z;\lambda) = (1,1,1;1)$ is one candidate point (there may be others)

Bordered Hessian: method to determine if an ordinary cand. pt is local max/min in the Lagrange problem.

$$J = (\,g'_x \quad g'_y \quad g'_z\,) = (2x \quad 2y \quad 2z)$$

$$H(\mathcal{L})_{\underline{x}} = \begin{pmatrix} 2y^2z^2 - 2\lambda & 4xyz^2 & 4xz\cdot y^2 \\ 4xyz^2 & 2x^2z^2 - 2\lambda & 4yzx^2 \\ 4xzy^2 & 4yz\cdot x^2 & 2x^2y^2 - 2\lambda \end{pmatrix}$$

$$B = \left(\begin{array}{c|c} 0 & J \\ \hline J^T & H(\mathcal{L})_{\underline{x}} \end{array}\right) = \left(\begin{array}{c|ccc} 0 & 2x & 2y & 2z \\ \hline 2x & & & \\ 2y & & H(\mathcal{L})_{\underline{x}} & \\ 2z & & & \end{array}\right)$$

$$B(1,1,1;1) = \begin{pmatrix} 0 & 2 & 2 & 2 \\ 2 & 0 & 4 & 4 \\ 2 & 4 & 0 & 4 \\ 2 & 4 & 4 & 0 \end{pmatrix}$$

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$$B(1,1,1;1) = \begin{pmatrix} 0 & 2 & 2 & 2 \\ 2 & 0 & 4 & 4 \\ 2 & 4 & 0 & 4 \\ 2 & 4 & 4 & 0 \end{pmatrix}$$

$(m+n) \times (m+n)$

$n=3$ # var's

$m=1$ # constr.

Compute the last $n-m$ leading principal minors:

$n-m = 3-1 = 2$: D_3, D_4

$$D_3 = \begin{vmatrix} 0 & 2 & 2 \\ 2 & 0 & 4 \\ 2 & 4 & 0 \end{vmatrix} = -2\cdot(-8) + 2\cdot 8 = 32 > 0$$

$$D_4 = \begin{vmatrix} 0 & 2 & 2 & 2 \\ 2 & 0 & 4 & 4 \\ 2 & 4 & 0 & 4 \\ 2 & 4 & 4 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 2 & 2 & 2 \\ 0 & -4 & 0 & 4 \\ 0 & 0 & -4 & 4 \\ 2 & 4 & 4 & 0 \end{vmatrix} = (-2)\cdot\left(2\cdot 16 + 4\cdot 16\right)$$

$$= -2\cdot(96) = -192 < 0$$

Signs alternating and the last sign is $(-1)^n = (-1)^3 = -1$ ok $\Rightarrow$ local max

Signs are constant and equal to sign of $(-1)^m = (-1) = -1$ not ok $\Rightarrow$ local min

Conclusion: $(x,y,z) = (1,1,1)$ is local max.

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Bordered Hessian: Kuhn-Tucker case

If $(x_1^*, x_2^*, \ldots, x_n^*; \lambda_1^*, \ldots, \lambda_m^*)$ is an ordinary candidate pt., we form matrix J:

$$J = \left(\partial g_i / \partial x_j \right)$$

We only include in J rows corresponding to binding constraints. Otherwise, the method is identical to the Lagrange case.