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LECTURE 8

GRA 6035 MATHEMATICS

EIVIND ERIKSEN OCT 13, 2016

Plan:

1. Constrained optimization
2. Second order conditions
3. Non-degenerate constraint qualification

Reading:

[MEJ 18.1-18.7, (12.3-12.5), 21.1, 19.1, 19.4,

1 Review: Constrained optimization

Lagrange problems	Kuhn-Tucker problems
max/min $f(\underline{x})$ when $\begin{cases} g_1(\underline{x}) = a_1 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases}$	max $f(\underline{x})$ when $\begin{cases} g_1(\underline{x}) \le a_1 \\ \vdots \\ g_m(\underline{x}) \le a_m \end{cases}$
	(allow min, allow $\geq$ but change them into the std. form above)
$\mathcal{L} = f(\underline{x}) - \lambda_1 g_1(\underline{x}) - \dots - \lambda_m g_m(\underline{x})$	$\mathcal{L} = f(\underline{x}) - \lambda_1 \cdot g_1(\underline{x}) - \dots - \lambda_m \cdot g_m(\underline{x})$
FOC: $\boxed{\begin{matrix} \mathcal{L}'_{x_1} = 0 \\ \vdots \\ \mathcal{L}'_{x_n} = 0 \end{matrix}}$ C: $\boxed{\begin{matrix} g_1(\underline{x}) = a_1 \\ \vdots \\ g_m(\underline{x}) = a_m \end{matrix}}$	FOC: $\boxed{\begin{matrix} \mathcal{L}'_{x_1} = 0 \\ \vdots \\ \mathcal{L}'_{x_n} = 0 \end{matrix}}$ C: $\boxed{\begin{matrix} g_1(\underline{x}) \le a_1 \\ \vdots \\ g_m(\underline{x}) \le a_m \end{matrix}}$
	CSC: $\boxed{\lambda_i \ge 0 \text{ and } \lambda_i \cdot (g_i(\underline{x}) - a_i) = 0 \text{ for } i = 1, 2, \dots, m.}$
$\underbrace{\qquad\qquad}_{\text{Lagrange conditions}}$	$\underbrace{\qquad\qquad}_{\text{Kuhn-Tucker conditions}}$
Solutions of the Lagrange conditions (FOC + C) = candidates for max/min	Solutions of the Kuhn-Tucker conditions (FOC + C + CSC) = candidates for max

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Ex: $\underline{\min\ x+3y}$ when $\underline{x^2+y^2 \le 10}$

not in std. form   std. form : ok.

$$= \boxed{\max\ -(x+3y) \text{ when } x^2+y^2 \le 10}\ \text{std. form.}$$

$$\mathcal{L} = -(x+3y) - \lambda \cdot (x^2+y^2)$$

FOC: $\mathcal{L}'_x = -1 - \lambda \cdot 2x = 0$

$\mathcal{L}'_y = -3 - \lambda \cdot 2y = 0$

C: $x^2+y^2 \le 10$

CSC: $\lambda \ge 0$

$\lambda \cdot (x^2+y^2-10) = 0$

$\Updownarrow$

$\lambda \ge 0$ if $x^2+y^2 = 10$ $\lambda = 0$ if $x^2+y^2 < 10$

a) $x^2+y^2 = 10$

$$\left.\begin{aligned} -1-\lambda\cdot 2x &= 0\\ -3-\lambda\cdot 2y &= 0\\ x^2+y^2 &= 10 \end{aligned}\right\}$$

$\lambda \ge 0$

$x = \dfrac{-1}{2\lambda}$ (ok since $\lambda \ne 0$)

$y = -\dfrac{3}{2\lambda}$

$$\left(\frac{-1}{2\lambda}\right)^2 + \left(\frac{-3}{2\lambda}\right)^2 = 10$$

$$\frac{1+9}{4\lambda^2} = 10$$

$$\frac{10}{4\lambda^2} = 10 \qquad 4\lambda^2 = 1$$

$\lambda^2 = 1/4$

$\lambda = \pm 1/2$

$\lambda = 1/2$

$$\boxed{x = -1,\ y = -3,\ \lambda = 1/2}$$

b) $x^2+y^2 < 10$

$-1-\lambda\cdot 2x = 0$

$-3-\lambda\cdot 2y = 0$

$x^2+y^2 < 10$

$\lambda = 0$

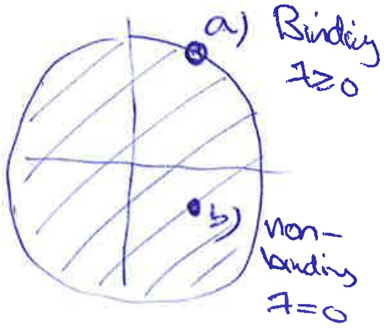
$\Downarrow$

$-1 = 0$

impossible

$\Downarrow$

no solutions



$x^2+y^2 \le 10$

admissible pts

$\lambda = 0$

$\Downarrow$

$\mathcal{L} = f(x) - \lambda \cdot g(x) = f(x)$

$\mathcal{L}'_x = f'_x = 0$

$\mathcal{L}'_y = f'_y = 0$

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Ex: $\min \overset{-10}{\overset{\shortparallel}{x+3y}}$ when $x^2+y^2 \le 10$ $\quad(-1,-3)$

$= \max \underset{\underset{10}{\shortparallel}}{\underline{-(x+3y)}}$ when $\underline{x^2+y^2 \le 10}$ $\quad(-1,-3)$

<u>Kuhn-Tucker, std. form:</u> FOC + C + CSC

$$(x,y;\lambda) = \underline{(-1,-3;\tfrac{1}{2})}$$

→ <u>Second order condition:</u>

$$\mathcal{L} = -(x+3y) - \lambda\cdot(x^2+y^2)$$

<u>Concave?</u> → $\mathcal{L}(x,y;\tfrac{1}{2}) = -x - 3y - \tfrac{1}{2}x^2 - \tfrac{1}{2}y^2$

$$\mathcal{L}'_x = -1 - x \qquad \mathcal{L}'_y = -3 - y$$

$$H(\mathcal{L}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad D_1 = -1, \quad D_2 = 1$$

~~pos.~~ neg. defn. for all $\underline{x}$

$\mathcal{L}(x,y;\tfrac{1}{2})$ is <u>concave</u>

$\Downarrow$

$(x,y) = (-1,-3)$ is <u>max</u> in the KT-pb.

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Thm (Necessary conditions in Lagrange / Kuhn-Tucker pb.)

If $\underline{x}^*$ is a optimal point (max/min) in a Lagrange/Kuhn-Tucker problem and $\underline{x}^*$ satisfies the NDCQ (non-degenerate constraint qualification), then

$(\underline{x}^*; \underline{\lambda}^*)$ satisfy FOC+C (Lagrange case) / FOC+C+CSC (Kuhn-tucker case)

$\underline{x}^*$ max/min $\xRightarrow[\text{holds}]{\text{NDCQ}}$ FOC+C / FOC+C+CSC must be satisfied

② Second order conditions

Let's say that $(\underline{x}^*; \underline{\lambda}^*)$ is a candidate for max/min that satisfies FOC+C (Lagrange case) / FOC+C+CSC (Kuhn-Tucker case) & Then:

Consider $H(\mathcal{L})$ where $\mathcal{L} = \mathcal{L}(x_1, \dots, x_n; \lambda_1^*, \lambda_2^*, \dots, \lambda_m^*)$: is a function in $(x_1, \dots, x_n)$.

If $\underbrace{H(\mathcal{L})(\underline{x}) \text{ is negative semidefinite for all } \underline{x}}_{\mathcal{L}(x_1, \dots, x_n; \lambda_1^*, \dots, \lambda_m^*) \text{ is concave}}$, then $\underline{x}^*$ is max.

If $\underbrace{H(\mathcal{L})(\underline{x}) \text{ is positive semidefinite for all } \underline{x}}_{\mathcal{L}(x_1, \dots, x_n; \lambda_1^*, \dots, \lambda_n^*) \text{ is convex}}$ then $\underline{x}^*$ is min.

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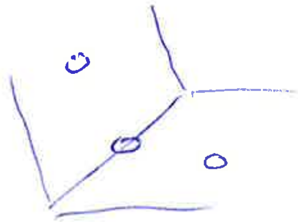
Ex: $\min\ 2x^2+y^2+3z^2$ when $\begin{cases} x-y+2z \geq 3 \\ x+y \geq 3 \end{cases}$

$= \max\ -(2x^2+y^2+3z^2)$ when

$-(x-y+2z) \leq -3$

$-(x+y) \leq -3$



$$\mathcal{L} = -2x^2-y^2-3z^2+\lambda_1(x-y+2z)+\lambda_2(x+y)$$

$$\boxed{\begin{array}{lll} \mathcal{L}'_x = -4x+\lambda_1+\lambda_2 = 0 & x-y+2z \geq 3 & \lambda_1 \geq 0,\ \lambda_2 \geq 0 \\ \mathcal{L}'_y = -2y-\lambda_1+\lambda_2 = 0 & x+y \geq 3 & \lambda_1\cdot(x-y+2z-3)=0 \\ \mathcal{L}'_z = -6z+2\lambda_1 = 0 & & \lambda_2\ (x+y-3)=0 \end{array}}$$

FOC

Case I: $\quad x-y+2z=3 \quad \lambda_1 \geq 0$
$\qquad\quad x+y = 3 \quad \lambda_2 \geq 0$

$-4x+\lambda_1+\lambda_2=0$

$-2y-\lambda_1+\lambda_2=0$

$-6z+2\lambda_1=0$

$x-y+2z=3$

$x+y=3$

$x = \frac{\lambda_1+\lambda_2}{4} = \frac{3\lambda_1+3\lambda_2}{12}$ $\quad \underline{x=2}$

$y = \frac{\lambda_2-\lambda_1}{2} = \frac{6\lambda_2-6\lambda_1}{12}$ $\quad \underline{y=1}$

$z = \frac{\lambda_1}{3} = \frac{4\lambda_1}{12}$ $\quad \underline{z=1}$

$\Downarrow$

$$\frac{(3\lambda_1+3\lambda_2)}{12} - \frac{(6\lambda_2-6\lambda_1)}{12} + \frac{2\cdot 4\lambda_1}{12} = 3$$

$$\boxed{17\lambda_1 - 3\lambda_2 = 36}$$

$$\frac{3\lambda_1+3\lambda_2}{12} + \frac{6\lambda_2-6\lambda_1}{12} = 3$$

$$\boxed{-3\lambda_1+9\lambda_2 = 36}$$

$3\cdot 17\lambda_1 - 3\lambda_1 = 4\cdot 36$

$\frac{48\lambda_1}{48} = \frac{\cancel{4}\cdot 36}{\cancel{48}\ 12}$

$\underline{\lambda_1 = 3} \quad \underline{\lambda_2 = 5}$

Conclusion: $(x,y,z;\lambda_1,\lambda_2) = \underline{(2,1,1;3,5)}$ candidates from Case I.

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$$\mathcal{L} = -2x^2 - y^2 - 3z^2 + \lambda_1(x-y+2z) + \lambda_2(x+y)$$

$$(x^*, y^*, z^*; \lambda_1^*, \lambda_2^*) = (2,1,1;3,5) \quad \text{candidate}$$

$$\mathcal{L}(x,y,z;3,5) = -2x^2 - y^2 - 3z^2 + \underbrace{3(x-y+2z) + 5(x+y)}_{\text{linear}}$$

$$H(\mathcal{L}) = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -6 \end{pmatrix} \quad \left.\begin{array}{l} D_1 = -4 \\ D_2 = 8 \\ D_3 = -48 \end{array}\right\} \text{neg. defn.}$$

$\mathcal{L}(x,y,z;3,5)$ concave

$\Downarrow$

$(x,y,z) = (2,1,1)$ is the <u>max</u>

Other methods for global max/min:

Extreme Value Thm:

If the set of admissible points is **bounded**, then the problem has <u>max/min</u>.

Need a list of all candidate pt's

Lagrange case:

i) Solutions of FOC+C
ii) Points where NDCQ fails

Kuhn–Tucker case:

i) Solutions of FOC+C+CSC
ii) Points where NDCQ fails

When we have a complete list, we compute $f(\underline{x})$ for each candidate point.

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③ **NDCQ:** Non-degenerate constraint qualification

Ex: $\underbrace{x^2+y^2}_{g(x,y)}=10$

Admissible points: $x^2+y^2=10$

NDCQ: $rk\begin{pmatrix} g'_x & g'_y \end{pmatrix}=1$

$rk\begin{pmatrix} 2x & 2y \end{pmatrix}=1$

NDCQ fails: $rk\begin{pmatrix} 2x & 2y \end{pmatrix}<1$

$\Updownarrow$

$2x=0\ ,\ 2y=0$

$(x,y)=(0,0)$ → not admissible

All admissible points satisfy NDCQ.

We look for admissible points where NDCQ fails

= candidate points.

Ex: $x^2+y^3=0$

Admissible: $x^2+y^3=0$

NDCQ: $rk\begin{pmatrix} 2x & 3y^2 \end{pmatrix}=1$

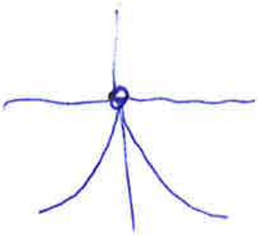
fails: $rk\begin{pmatrix} 2x & 3y^2 \end{pmatrix}<1$

$2x=0 \quad 3y^2=0$

$(x,y)=(0,0)$ → ok

Admissible pts where NDCQ fails: $(0,0)$

Ex: max y when $x^2+y^3=0$



$\mathcal{L}=y-\lambda\cdot(x^2+y^3)$

$$\boxed{\begin{aligned} \mathcal{L}'_x &= -\lambda\cdot 2x=0 \\ \mathcal{L}'_y &= 1-\lambda\cdot 3y^2=0 \\ & x^2+y^3=0 \end{aligned}}$$

$\lambda=0$ / or $x=0$

$\cancel{1=0}$ impossible

$\cancel{1=0}$, $\cancel{y=0}$ impossible

NDCQ.

$\Rightarrow$ no ordinary candidate pts.

special cand. pt. $(0,0)$

$\Downarrow$

max

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General NDCQ: Lagrange case

$$\boxed{\begin{aligned} g_1(\underline{x}) &= a_1 \\ &\vdots \\ g_m(\underline{x}) &= a_m \end{aligned}}$$

constraints

$$\text{rk}\begin{pmatrix} \partial g_1/\partial x_1 & \partial g_1/\partial x_2 & \cdots & \partial g_1/\partial x_n \\ \partial g_2/\partial x_1 & \partial g/\partial x_2 & \cdots & \partial g_2/\partial x_n \\ \vdots & & & \\ \partial g_m/\partial x_1 & \partial g_m/\partial x_1 & \cdots & \partial g_m/\partial x_n \end{pmatrix} = m$$

Ex: $x - y + 2z = 3$, $x + y = 3$ → NDCQ: $\text{rk}\begin{pmatrix} \boxed{\begin{matrix} 1 & -1 \\ 1 & 1 \end{matrix}} & \begin{matrix} 2 \\ 0 \end{matrix} \end{pmatrix} = 2$ ok

$\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1 + 1 = 2 \neq 0$

$\text{rk} = 2$

No adm. pts where NDCQ fails.

General NDCQ: Kuhn-Tucker case

$$\boxed{\begin{aligned} g_1(\underline{x}) &\le a_1 \\ &\vdots \\ g_m(\underline{x}) &\le a_m \end{aligned}}$$

$$\text{rk}\begin{pmatrix} \partial g_1/\partial x_1 & \cdots & \partial g_1/\partial x_n \\ \vdots & & \\ \partial g_m/\partial x_1 & \cdots & \partial g_m/\partial x_n \end{pmatrix} = \#\text{rows}$$

include only rows corresponding to binding constraint $g_i(\underline{x}) = a_i$

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Ex: $\boxed{\begin{aligned} x-y+2z &\geq 3 \\ x+y &\geq 3 \end{aligned}}$ Constraints in K-T problem

Check NDCQ:

a) $\left.\begin{aligned} x-y+2z &= 3 \\ x+y &= 3 \end{aligned}\right\}$ $\text{rk}\begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & 0 \end{pmatrix} = 2$ ok.

b) $\left.\begin{aligned} x-y+2z &= 3 \\ x+y &> 3 \end{aligned}\right\}$ $\text{rk}\begin{pmatrix} 1 & -1 & 2 \end{pmatrix} = 1$ ok.

c) $\left.\begin{aligned} x-y+2z &> 3 \\ x+y &= 3 \end{aligned}\right\}$ $\text{rk}\begin{pmatrix} 1 & 1 & 0 \end{pmatrix} = 1$ ok.

d) $\left.\begin{aligned} x-y+2z &> 3 \\ x+y &> 3 \end{aligned}\right\}$ no condition ok

in each case, include only rows corresp. to binding constraints

No adm. pts where NDCQ fails.

Overview: Methods

① Find usual candidate pts
(FOC/C FOC/C/CSC)

② Use second order condition (SOC) if possible. (convex / concave)

③ Use extrem value thm. if possible (bounded set)

④ If none of the methods ②,③ work, try something else!

Ex: min $2x^2+y^2+3z^2$ when $x-y+2z \geq 3$
$\qquad\qquad\qquad\qquad x+y \geq 3$

$f(x,y,z)$

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Std. form

$$\boxed{\begin{array}{lll} \max \; -f(x,y,z) & \text{when} & -x+y-2z \leq -3 \\ \quad -2x^2-y^2-3z^2 & & -x-y \leq -3 \end{array}}$$

$$\mathcal{L} = -2x^2-y^2-3z^2-\lambda_1(-x+y-2z)-\lambda_2\cdot(-x-y)$$

KT Conditions

FOC:
$$\mathcal{L}'_x = -4x + \lambda_1 + \lambda_2 = 0$$
$$\mathcal{L}'_y = -2y - \lambda_1 + \lambda_2 = 0$$
$$\mathcal{L}'_z = -6z + 2\lambda_1 = 0$$

C:
$$x-y+2z \geq 3$$
$$x+y \geq 3$$

CSC:
$\lambda_1 \geq 0$ and $\lambda_1\cdot(x-y+2z-3)=0$
$\lambda_2 \geq 0$ and $\lambda_2\cdot(x+y-3)=0$

$x-y+2z=3$, $x+y=3$	$x-y+2z=3$, $x+y>3$	$x-y+2z>3$, $x+y=3$	$x-y+2z>3$, $x+y>3$
$\lambda_1 \geq 0$, $\lambda_2 \geq 0$	$\lambda_1 \geq 0$, $\lambda_2 = 0$	$\lambda_1 = 0$, $\lambda_2 \geq 0$	$\lambda_1 = 0$, $\lambda_2 = 0$
FOC	FOC	FOC	FOC
	$\lambda_1 = 4x$, $\lambda_1 = -2y$, $\lambda_1 = 3z$; $x = \frac{\lambda_1}{4}$, $y = -\frac{\lambda_1}{2}$; $z = \frac{\lambda_1}{3}$; $\frac{\lambda_1}{4} + \frac{\lambda_1}{2} + \frac{2}{3}\lambda_1 = 3$; $17\lambda_1 = 36$	$z=0$; $\lambda_2 = 4x = 2y$; $\Rightarrow y = 2x$; $x+y=3$; $x+2x=3$; $x=1, y=2, z=0, \lambda_1=0, \lambda_2=4$ not admissible	$x=y=z=0$ not admissible

$\lambda_1 = \frac{36}{17}$

$x = \frac{36}{68} \quad y = -\frac{36}{34}$

$x+y = \frac{36}{68} - \frac{72}{68} = \frac{-36}{68}$

not admissible

$x - y + 2z = 3$

$x + y = 3$

$\lambda_1 \geq 0$

$\lambda_2 \geq 0$

$-4x + \lambda_1 + \lambda_2 = 0$

$-2y - \lambda_1 + \lambda_2 = 0$

$-6z + 2\lambda_1 = 0$

$\frac{\lambda_1+\lambda_2}{4} - \frac{-\lambda_1+\lambda_2}{2} + 2 \cdot \frac{\lambda_1}{3} = 3 \quad | \cdot 12$

$\frac{\lambda_1+\lambda_2}{4} + \frac{-\lambda_1+\lambda_2}{2} = 3 \quad | \cdot 12$

$x = \frac{\lambda_1+\lambda_2}{4}$

$y = \frac{-\lambda_1+\lambda_2}{2}$

$z = \frac{\lambda_1}{3}$

$3(\lambda_1+\lambda_2) - 6(-\lambda_1+\lambda_2) + 8\lambda_1 = 36$

$3(\lambda_1+\lambda_2) + 6(-\lambda_1+\lambda_2) = 36$

$17\lambda_1 - 3\lambda_2 = 36$

$-3\lambda_1 + 9\lambda_2 = 36$

$48\lambda_1 = 144$

$\lambda_1 = \frac{144}{48} = \frac{12}{4} = 3$

$-9 + 9\lambda_2 = 36$

$\lambda_2 = \frac{45}{9} = 5$

Adm. pt: $(2, 1, 1; 3, 5)$ $\quad -f = -12$ $\quad (f = 12)$