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LECTURE 6

GRA 6035

EIVIND ERIKSEN

SEP 29, 2016

MATHEMATICS

Plan:

1. Review: Definiteness of quadratic forms
2. Unconstrained optimization
3. Convex and concave functions

Reading:

[MEJ] 14.1–14.4, 14.8, 17.1–17.5

Midterm exam: Next Friday

Lecture 7 – 8

Tue at 17–20: (next week)

Please tell me if there are problems you want me to do.

Workbook 1–6 + Exam problems

1. Review:

Ex: $Q(x_1,x_2,x_3,x_4) = -x_1^2 + 4x_1x_2 + 3x_1x_3 - 5x_2^2 - 6x_3^2 - x_4^2$

$$A = \begin{pmatrix} -1 & 2 & 3/2 & 0 \\ 2 & -5 & 0 & 0 \\ 3/2 & 0 & -6 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

4x4, symmetric

1. Defn: $Q(\underline{x}) = \underline{x}^t \cdot A \cdot \underline{x}$
2. Signs of eigenvalues of A: $\lambda_1, \dots, \lambda_4$.
3. Principal minors of A and their signs

Indefinite $\longrightarrow$ leading principal minors

$$\begin{cases} D_1 = -1 < 0 \\ D_2 = 5 - 4 = 1 > 0 \\ D_3 = 3/2 \cdot 15/2 + (-6) \cdot 1 = 45/4 - 6 = \frac{21}{4} > 0 \\ D_4 = -1 \cdot \frac{21}{4} = -\frac{21}{4} < 0 \end{cases}$$

Criterion:

Main

$D_1, D_2, D_3, \ldots, D_n > 0 \iff Q$ positive definite

$\left.\begin{array}{l} D_1, D_3, D_5, \ldots < 0 \text{ and} \\ D_2, D_4, \ldots > 0 \end{array}\right\} \iff Q$ negative definite

Next

$\left.\begin{array}{l} \Delta_1, \Delta_2, \ldots, \Delta_n \geq 0 \\ \text{for all principal minors} \end{array}\right\} \iff Q$ positive semidefinite

$\left.\begin{array}{l} \Delta_1, \Delta_3, \Delta_5, \ldots \leq 0 \\ \Delta_2, \Delta_4, \Delta_6, \ldots \geq 0 \\ \text{for all principal minors} \end{array}\right\} \iff Q$ negative semidefinite

all other cases $\iff Q$ indefinite

Typical examples:

$D_2 < 0$ $(D_4 < 0, D_6 < 0, \ldots) \implies Q$ indefinite

D_1 and D_3 have opposite signs $\implies Q$ indefinite

(or two odd order principal minors)

2 Unconstrained optimization

Functions in one and two variables
Calculus (derivatives)
Max/min

} FORK1003: Lecture 4-6
EMEA: First chapters

$f(x_1, x_2, x_3, \ldots, x_n) = f(\underline{x})$ $\qquad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

function in many variables

Unconstrained optimization:

$$\max / \min \; f(\underline{x}) = f(x_1 \ldots x_n)$$

Ex: $f(x,y,z) = \underbrace{x^2+y^2-3yz+z^2}_{\text{quadratic form}} \underbrace{-2x+y-z}_{\text{degree 1 linear form}} \underbrace{+1}_{\text{constant}}$

Partial derivatives:

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= f'_x = \underline{2x-2} \\ \frac{\partial f}{\partial y} &= f'_y = \underline{2y-3z+1} \\ \frac{\partial f}{\partial z} &= f'_z = \underline{-3y+2z-1} \end{aligned}\right\} \qquad H(f) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & -3 \\ 0 & -3 & 2 \end{pmatrix}$$

first order partial derivatives

Hessian matrix of f
(second order partial derivatives)

$$H(f) = \begin{pmatrix} f''_{xx} & f''_{xy} & f''_{xz} \\ f''_{yx} & f''_{yy} & f''_{yz} \\ f''_{zx} & f''_{zy} & f''_{zz} \end{pmatrix}$$

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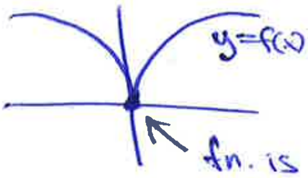
Defn:

f is called a C^2-function if all the second order partial derivatives of f exists and are continuous.

Ex: $f(x) = x^{2/3} = \sqrt[3]{x^2}$, $x \geq 0$

$f'(x) = \frac{2}{3} x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$, $x \neq 0$



fn. is defined but it is not diff.

$f(0) = 0$

$f'(0)$ is not defined

We only consider C^2-functions in this course

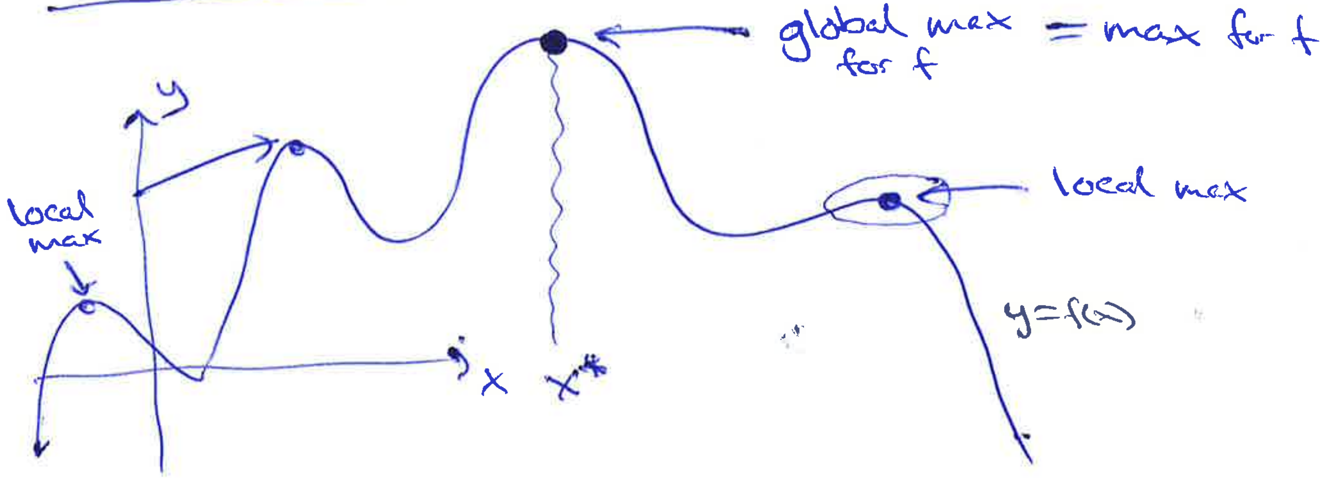
Fact: $H(f)$ is symmetric (when f is C^2)

$$H(f) = \begin{pmatrix} f''_{11} & f''_{12} & \cdots & f''_{1n} \\ f''_{21} & f''_{22} & \cdots & f''_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f''_{n1} & f''_{n2} & \cdots & f''_{nn} \end{pmatrix}$$

symmetric $\Longleftrightarrow$ $f''_{ij} = f''_{ji}$

Ex: $H(f) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & -3 \\ 0 & -3 & 2 \end{pmatrix}$

Local / global max / min for $f(\underline{x})$



$\underline{x}^*$ global max = max for f : $f(\underline{x}^*) \geq f(\underline{x})$ for all $\underline{x}$ in D_f

global min = min " $f(\underline{x}^*) \leq f(\underline{x})$ — " —

$\underline{x}^*$ local max for f : $f(\underline{x}^*) \geq f(\underline{x})$ for all $\underline{x}$ nearby

local min " $f(\underline{x}^*) \leq f(\underline{x})$ — " —

Method for finding unconstrained global max/min

$f(x_1, \dots, x_n) = f(\underline{x})$:

Ex: $f(x,y) = x^3 + 3xy - y^3$

Defn. A stationary point for f is a point where

$$\underbrace{f'_{x_1} = f'_{x_2} = \dots = f'_{x_n} = 0}_{\text{FOC = first order conditions}}.$$

$$f'_x = 3x^2 + 3y = 0$$
$$f'_y = 3x - 3y^2 = 0$$

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$f'_x = 3x^2 + 3y = 0$ $\qquad x^2 + y = 0$ $\qquad (y^2)^2 + y = 0$

$f'_y = 3x - 3y^2 = 0$ $\qquad x - y^2 = 0 \Rightarrow \underline{x = y^2}$

$y^4 + y = 0$

$y(y^3 + 1) = 0$

$\underline{y = 0}$ or $\underline{y = -1} = \sqrt[3]{-1}$

$x = 0 \qquad x = 1$

Stationary pts

$(x, y) = (0,0)$

$(1, -1)$

candidates for max/min

Fact:
If $\underline{x}^*$ is a max/min for f, then it is a stationary pt.

$\underline{x}^*$ max/min for $f \Longrightarrow \underline{x}^*$ stationary pt.

Classify stationary pts as local max, local min, saddle pt.

Defn: Saddle pt = stationary pt that is not local max and not local min.

Fact: Second derivative test
Look at $H(f)(\underline{x}^*)$ for a stationary pt. $\underline{x}^*$.

$H(f)(\underline{x}^*)$ pos. definite $\Longrightarrow \underline{x}^*$ local min

$H(f)(\underline{x}^*)$ neg. definite $\Longrightarrow \underline{x}^*$ local max

$H(f)(\underline{x}^*)$ indefinite $\Longrightarrow \underline{x}^*$ saddle pt

Otherwise, the test is inconclusive.

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Ex: $f = x^3 + 3xy - y^3$

$f'_x = 3x^2 + 3y$

$f'_y = 3x - 3y^2$

Stationary pts:

$(x,y) = (0,0)$

$(1,-1)$

Method:

- Find all stationary pts

$f'_{x_1} = f'_{x_2} = \dots = f'_{x_n} = 0$

- Classify them as local max/min, saddle pt.

$H(f)(x^*)$

$f(1,-1) = -1 \qquad f(0,2) = -8$

$$H(f)(x,y) = \begin{pmatrix} 6x & 3 \\ 3 & -6y \end{pmatrix}$$

At $(0,0)$:

$$H(f)(0,0) = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

$$\left.\begin{aligned} A &= D_1 = 0 \\ AC - B^2 &= D_2 = -9 \end{aligned}\right\} \text{indefinite}$$

$\underbrace{A > 0, \ AC - B^2 > 0}_{D_1,\ D_2}$: local min

$(0,0)$ saddle pt

At $(1,-1)$:

$$H(f)(1,-1) = \begin{pmatrix} 6 & 3 \\ 3 & 6 \end{pmatrix}$$

$$\left.\begin{aligned} D_1 &= 6 \\ D_2 &= 36 - 9 = 27 \end{aligned}\right\} \text{pos. definite}$$

$(1,-1)$ local min

$\Downarrow$

Conclusion:

No max.

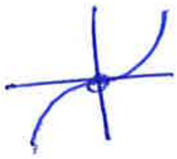
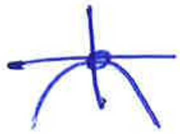
Candidate for min: $(1,-1)$

$f(1,-1) = -1$

No global min since $f(0,2) = -8 < f(1,-1) = -1$

Ex:

$f = x^4$ | $f = x^3$ | $f = -x^4$

$f' = 4x^3$ $f''(0) = 0$ local min

$f' = 3x^2$ $f''(0) = 0$ saddle pt

$f' = -4x^3$ $f''(0) = 0$ local max

$H(f)(0) = 0$:
pos/neg. semidef.
but not pos/neg.
definite

Ex: $f(x,y,z) = x^3 + y^3 - z^3$

$$\left.\begin{aligned} f'_x &= 3x^2 = 0 \\ f'_y &= 3y^2 = 0 \\ f'_z &= -3z^2 = 0 \end{aligned}\right\} \quad (x,y,z) = (0,0,0) \text{ stationary pt.}$$

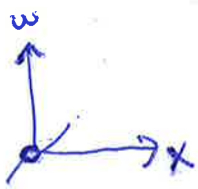
$$H(f) = \begin{pmatrix} 6x & 0 & 0 \\ 0 & 6y & 0 \\ 0 & 0 & -6z \end{pmatrix}$$

$$H(f)(0,0,0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

pos./neg. semidefinite
inconclusive test

Classification without the test:

$f(0,0,0) = 0$ $f(a,0,0) = a^3$ $\begin{cases} > 0 & \text{if } a > 0 \\ < 0 & \text{if } a < 0 \end{cases}$



← saddle pt.

An Example

Let $f(x,y) = x^2y^3 + y^2 - 2y$. Then f has just <u>one</u> stationary pt, which is a <u>local min</u>. However, f has <u>no global min</u>.

$$\left.\begin{aligned} f'_x &= 2xy^3 = 0 \\ f'_y &= 3x^2y^2 + 2y - 2 = 0 \end{aligned}\right\} \text{ FOC:}$$

$x = 0$ or $y = 0$

If $x = 0$, then (2) gives

$2y - 2 = 0 \Rightarrow \underline{y = 1}$

If $y = 0$, then (2) gives

$-2 = 0$ <u>no solution</u>

$$H(f) = \begin{pmatrix} 2y^3 & 6xy^2 \\ 6xy^2 & 6x^2y + 2 \end{pmatrix}$$

<u>Conclusion</u>: $(0,1)$ is the only stationary pt, with $\underline{f(0,1) = -1}$

$$H(f)(0,1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad \begin{aligned} D_1 &= 2 > 0 \\ D_2 &= 4 > 0 \end{aligned}$$

pos. defn. $\Rightarrow$ $(0,1)$ <u>local min</u>

But $(x,y) = (0,1)$ with $f(0,1) = -1$ is <u>not</u> global min

For example,

$$f(3,-1) = -9 + 1 + 2 = -6 < -1$$

<u>Conclusion:</u> f has no global min even if $(0,1)$ is local min and this is the only stationary point.

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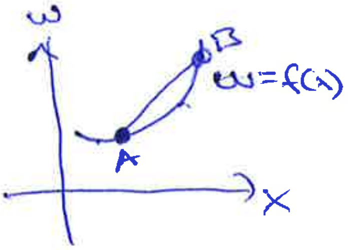
3 Convex / concave functions

Defn: $f(x_1, \ldots, x_n)$ is convex

$\Updownarrow$

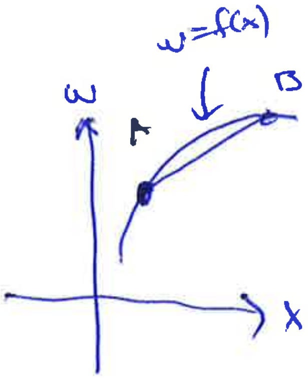
For any pts A and B on the graph of f, the line segment $[A, B]$ lies over the graph of f (or on the graph)



$f(x_1, \ldots, x_n)$ is concave

$\Updownarrow$

For any pts A and B on the graph of f, the line segment $[A, B]$ lies under the graph of f (or on the graph)



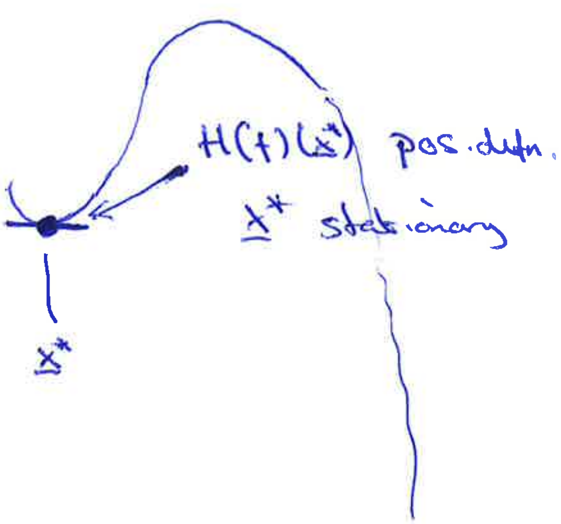
Test:

f convex $\Longleftrightarrow$ $H(f)$ is positive semidefinite for all $\underline{x}$ in D_f.
$\quad\quad\quad\quad\quad\quad\; \| $
$\quad\quad\quad\quad\quad H(f)(\underline{x})$

f concave $\Longleftrightarrow$ $H(f)$ is negative semidefinite for all $\underline{x}$ in D_f
$\quad\quad\quad\quad\quad\quad\; \| $
$\quad\quad\quad\quad\quad H(f)(\underline{x})$

Result:

If f is convex, then any stationary pt is a global min.
If f is concave, then — " — is a global max.



H(f) is pos semidefn. at all pts

Ex: $f = x^3 + 3xy - y^3$ convex/concave?

$f'_x = 3x^2 + 3y$

$f'_y = 3x - 3y^2$ $\Rightarrow$ $H(f) = \begin{pmatrix} 6x & 3 \\ 3 & -6y \end{pmatrix}$

$D_1 = 6x$

$D_2 = -36xy - 9$

f convex

$D_1 \geq 0$ for all (x,y)

$6x \geq 0$ for all (x,y) $\leftarrow$ not satisfied

f not convex

not concave

$D_2 \geq 0$ — " —

$-36xy - 9 \geq 0$ — " —

Ex: $f = e^{2x-y}$ is convex

$f'_x = e^{2x-y} \cdot 2 = 2e^{2x-y}$

$f'_y = e^{2x-y} \cdot (-1) = -e^{2x-y}$

$H(f) = \begin{pmatrix} 4e^{2x-y} & -2e^{2x-y} \\ -2e^{2x-y} & e^{2x-y} \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \cdot e^{2x-y}$

$D_1 = 4e^{2x-y} > 0$ for all x, y $\qquad \Delta_1 = D_1, e^{2x-y} > 0$

$D_2 = 4(e^{2x-y})^2 - 4(e^{2x-y})^2 = 0 \geq 0$ for all x, y

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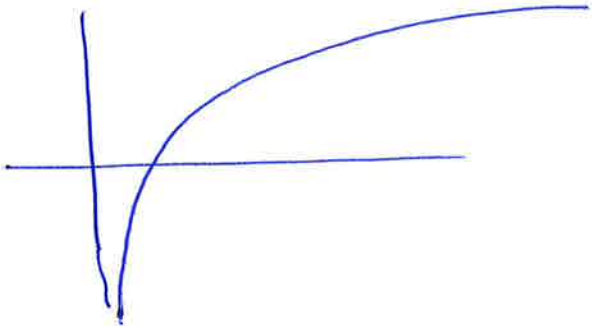
Not all convex/concave fn's have global max/min:

Ex: $f(x) = \ln(x), \; x>0$

$f'(x) = \frac{1}{x}$

$f'' = -\frac{1}{x^2} < 0, \; x>0$

Concave



no global max since there is no stationary pt.

$\left(\begin{array}{c} f' = \frac{1}{x} = 0 \\ \text{no solution} \end{array} \right)$