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LECTURE 5

GRA 6035 MATHEMATICS

EIVIND ERIKSEN, SEP 22 2016

Plan:

1. Diagonalization and Markov chains
2. Quadratic forms, definiteness and principal minors.

Reading:

[ME] 6.2 (Ex 3), 23.1 (Ex 23.4), 23.6, 13.1 – 13.5, 16.1 – 16.4, 23.8

Review:

A ($n \times n$-matrix) is <u>diagonalizable</u> if there is an invertible matrix P and a diagonal matrix D s.t.

$$P^{-1}AP = D$$

A <u>diagonalizable</u> $\iff$ i) there are n <u>eigenvalues</u> of A $\lambda_1, \lambda_2, \ldots, \lambda_n$ (counted with multiplicity) and ii) there are n linearly independent <u>eigenvectors</u> $\underline{v_1}, \underline{v_2}, \ldots, \underline{v_n}$

$$D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

$$P = \left(\underline{v_1} \,\middle|\, \underline{v_2} \,\middle|\, \ldots \,\middle|\, \underline{v_n} \right)$$

$\underline{v_i}$ eigenvector for λ_i

<u>Fact:</u> If λ is an eigenvalue with multiplicity m, then

$$1 \leq \underbrace{\dim E_\lambda}_{\text{the \# of lin. independent eigenvectors for } \lambda} \leq m$$

Condition ii) $\Longleftrightarrow$ For each eigenvalue with multiplicity m, we have $\dim E_\lambda = m$.

Facts:

i) If A is symmetric $(A^T = A)$, then A is diagonalizable.

ii) If A has n distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (all multiplicities are 1), then A is diagonalizable.

If not, we must compute all eigenvalues / -vectors and check i) - ii).

① Applications: Markov Chains

Ex: employment

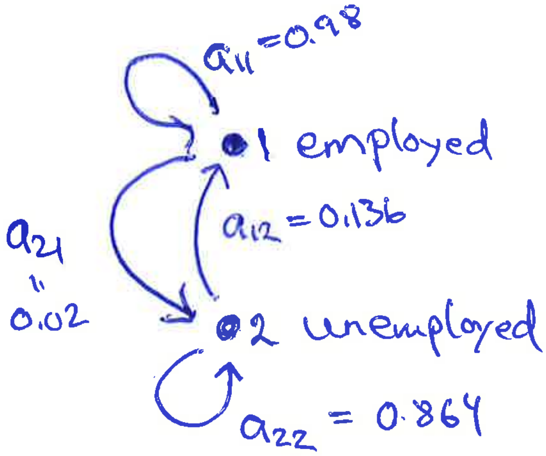
State vector $\underline{x}_t = \begin{pmatrix} e_t \\ u_t \end{pmatrix}$

$$\begin{cases} e_t \geq 0,\ u_t \geq 0 \\ e_t + u_t = 1 \end{cases}$$

$$\underline{x}_{t+1} = \underbrace{A}_{\substack{2\times 2\text{-matrix} \\ \text{transition matrix}}} \cdot \underline{x}_t$$

$a_{ij} \geq 0$
column sums $= 1$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0.98 & 0.136 \\ 0.02 & 0.864 \end{pmatrix}$$



a_{ij}: transitions from node j to node i

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$\underline{x_{t+1}} = A \cdot \underline{x_t}$

$$\begin{pmatrix} e_{t+1} \\ u_{t+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} e_t \\ u_t \end{pmatrix}$$

$$e_{t+1} = a_{11} e_t + a_{12} u_t$$
$$u_{t+1} = a_{21} e_t + a_{22} u_t$$

$$\underline{x_0} \longrightarrow \underline{x_1} = A \cdot \underline{x_0} \longrightarrow \underline{x_2} = A \cdot \underline{x_1} \longrightarrow \dots \longrightarrow \underline{x_n} = A^n \cdot \underline{x_0}$$
$$= A^2 \cdot \underline{x_0}$$

Target: Find $\lim_{n \to \infty} A^n \cdot x_0$

① Find eigenvalues and eigenvectors of A

$$A = \begin{pmatrix} 0.98 & 0.136 \\ 0.02 & 0.864 \end{pmatrix}$$

$$\begin{vmatrix} 0.98-\lambda & 0.136 \\ 0.02 & 0.864-\lambda \end{vmatrix} = 0 \qquad 0.98 \cdot 0.864 - 0.02 \cdot 0.136$$

$\boxed{\lambda_1 \neq \lambda_2 : A \text{ is diagonalizable}}$

$$\lambda^2 - 1.844\lambda + 0.844 = 0$$
$$\lambda_1 = 1, \quad \lambda_2 = 0.844 \rightsquigarrow D = \begin{pmatrix} 1 & 0 \\ 0 & 0.844 \end{pmatrix}$$

$$P^{-1}AP = D \Rightarrow A = PDP^{-1}$$
$$\underline{A^n = P \cdot D^n \cdot P^{-1}}$$

$$D^n = \begin{pmatrix} 1 & 0 \\ 0 & 0.844^n \end{pmatrix} \xrightarrow{n \to \infty} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$\underline{\lambda=1}$: $\begin{pmatrix} -0.02 & 0.136 \\ \cancel{0.02} & \cancel{-0.136} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ \cancel{0} \end{pmatrix}$

$$0.02x = 0.136y$$
$$x = 6.8y \rightarrow E_1 : \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6.8y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} 6.8 \\ 1 \end{pmatrix}$$

$\underline{\lambda = 0.844}$: $\begin{pmatrix} 0.136 & 0.136 \\ \cancel{0.02} & \cancel{0.02} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad x+y=0$, $x=-y$

$$E_{\lambda_2} : \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

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$$P = \begin{pmatrix} 68 & -1 \\ 10 & 1 \end{pmatrix} \qquad P^{-1} = \frac{1}{78}\begin{pmatrix} 1 & 1 \\ -10 & 68 \end{pmatrix}$$

$$A^n = P \cdot D^n \cdot P^{-1} = \begin{pmatrix} 68 & -1 \\ 10 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0.844^n \end{pmatrix} \cdot \frac{1}{78}\begin{pmatrix} 1 & 1 \\ -10 & 68 \end{pmatrix}$$

$$\Big\downarrow n \to \infty$$

$$\begin{pmatrix} 68 & -1 \\ 10 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \frac{1}{78}\begin{pmatrix} 1 & 1 \\ -10 & 68 \end{pmatrix}$$

$$= \frac{1}{78}\begin{pmatrix} 68 & 0 \\ 10 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ -10 & 68 \end{pmatrix}$$

$$= \frac{1}{78}\begin{pmatrix} 68 & 68 \\ 10 & 10 \end{pmatrix} = \begin{pmatrix} 68/78 & 68/78 \\ 10/78 & 10/78 \end{pmatrix}$$

$$\lim_{n\to\infty} (A^n \cdot \underline{x_0}) = \begin{pmatrix} 68/78 & 68/78 \\ 10/78 & 10/78 \end{pmatrix} \cdot \begin{pmatrix} e_0 \\ u_0 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 68/78 \\ 10/78 \end{pmatrix}}} \simeq \begin{pmatrix} 0.872 \\ 0.128 \end{pmatrix}$$

$$\| $$

$$e_0 \cdot \begin{pmatrix} 68/78 \\ 10/78 \end{pmatrix} + u_0 \cdot \begin{pmatrix} 68/78 \\ 10/78 \end{pmatrix}$$

employment/unemploym. rate in the long run.

Equilibrium state: $\underline{x} = \lim_{n\to\infty} (A^n \cdot x_0)$

$$E_1 : y \cdot \begin{pmatrix} 6.8 \\ 1 \end{pmatrix}$$

State vector: $y \cdot 6.8 + y \cdot 1 = 1$

$$y \cdot 7.8 = 1 \Rightarrow y = \frac{1}{7.8}$$

$$\underline{v} = \begin{pmatrix} 6.8/7.8 \\ 1/7.8 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 68/78 \\ 10/78 \end{pmatrix}}}$$

Results for Markov chains

<u>n categories</u> : state vector (n-vector) $\underline{x}_t = \begin{pmatrix} x_t(1) \\ x_t(2) \\ \vdots \\ x_t(n) \end{pmatrix}$

$$\begin{cases} x_t(i) \geq 0 \\ \sum_i x_t(i) = 1 \end{cases}$$

transition matrix $A = (a_{ij})$ n×n-matrix

$$\boxed{\underline{x}_{t+1} = A \cdot \underline{x}_t}$$

$$\begin{cases} a_{ij} \geq 0 \quad \text{for all } i,j \\ \text{col. sums} = 1 \end{cases}$$

The Markov chain is called <u>regular</u> if you can get to any node from any other node.

$$\boxed{a_{ij} > 0 \text{ for all } i,j \implies A \text{ is regular}}$$

<u>Result:</u> Assume that the Markov chain is regular. Then we have:

i) $\lambda = 1$ is an <u>eigenvalue</u>, and it is <u>dominant</u> ($|\lambda_i| < 1$ for all other eigenvalues)

ii) there is a unique state vector among the eigenvectors for $\lambda = 1$, $\underline{v}$.

iii) For any initial state $\underline{x}_0$, we have that

$$\lim_{t\to\infty} \underline{x}_t = \lim_{t\to\infty} A^t \cdot \underline{x}_0 = \underline{v}$$

Markov process

Ex: Families are classified as U (urban), S (suburban) and R (rural). At time $t=n$ (after n years), the share of families in these groups can be described by the state vector

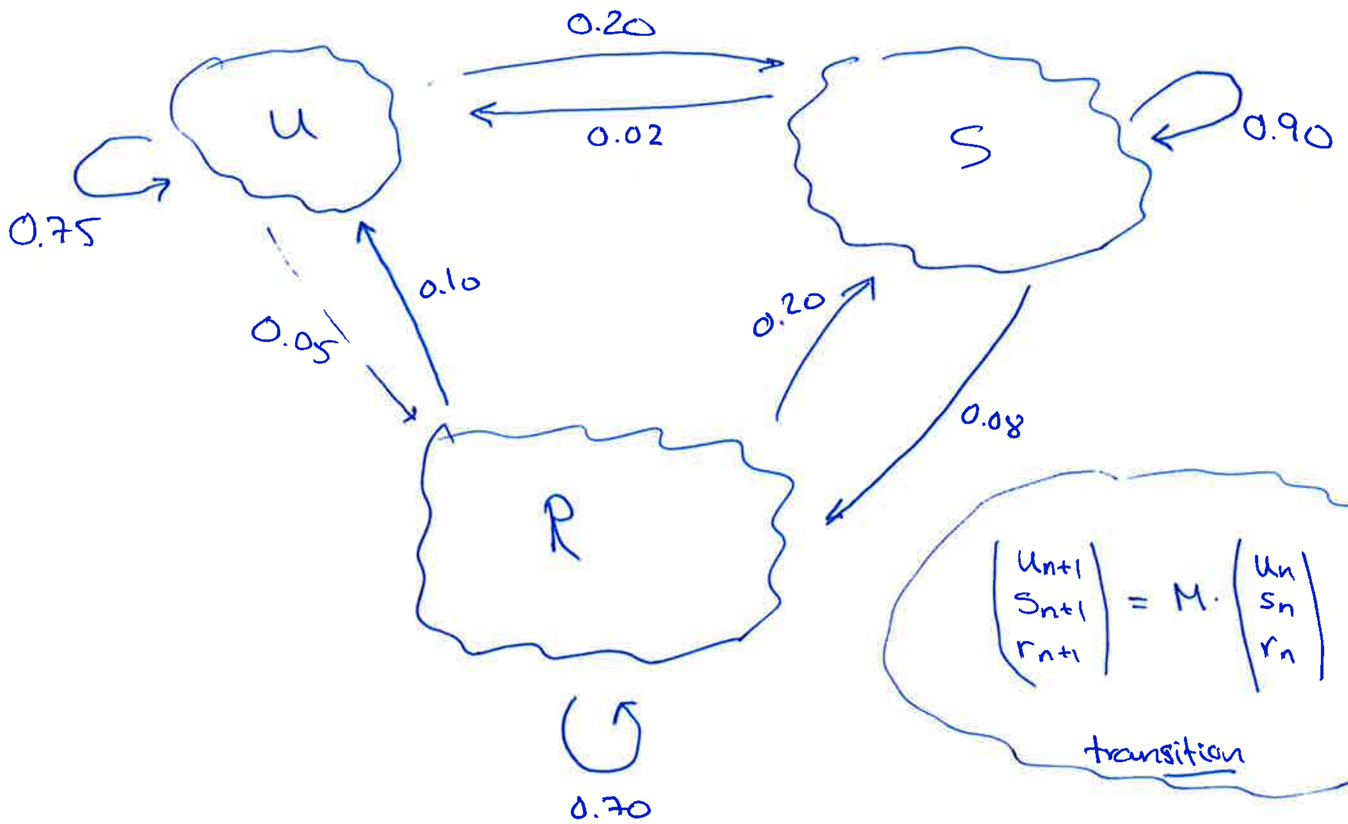
$$\underline{V}_n = \begin{pmatrix} u_n \\ s_n \\ r_n \end{pmatrix} \quad \begin{cases} u_n \geq 0 \\ s_n \geq 0 \\ r_n \geq 0 \end{cases}, \quad u_n + s_n + r_n = 1$$

Ex: $\underline{V} = \begin{pmatrix} 0.8 \\ 0.1 \\ 0.1 \end{pmatrix}$

From year n to year n+1, the change in the shares are given by a transition matrix or Markov matrix

$$M = \begin{pmatrix} 0.75 & 0.02 & 0.10 \\ 0.20 & 0.90 & 0.20 \\ 0.05 & 0.08 & 0.70 \end{pmatrix} \quad \begin{cases} m_{ij} \geq 0 \\ \text{each column has sum } 1 \end{cases}$$

It can be described graphically as follows:



$$\begin{pmatrix} u_{n+1} \\ s_{n+1} \\ r_{n+1} \end{pmatrix} = M \cdot \begin{pmatrix} u_n \\ s_n \\ r_n \end{pmatrix}$$

transition

Markov process:

$$\underline{V}_0 = \begin{pmatrix} u_0 \\ s_0 \\ r_0 \end{pmatrix} \dashrightarrow \underline{V}_1 = M \cdot \underline{V}_0 \dashrightarrow \underline{V}_2 = M\underline{V}_1 = M^2 \underline{V}_0 \dashrightarrow \cdots \longrightarrow \underline{V}_n = M^n \cdot \underline{V}_0$$

start

The Markov process is <u>regular</u> if $m_{ij} > 0$ for all i,j. We assume that this the case. The following holds for all regular Markov processes:

<u>Fact:</u> i) $\lambda = 1$ is an eigenvalue of M, and there is a unique eigenvector $\underline{v}$ with eigenvalue $\lambda = 1$ that is a state vector (i.e. $\underline{v} = (v_i)$ with $v_i \geq 0$, $v_1 + \dots + v_k = 1$)

ii) $\lim_{n\to\infty} M^n \cdot \underline{v}_0 = \underline{v}$ and $\lim_{n\to\infty} M^n = (\underline{v} | \underline{v} | \dots | \underline{v})$

<u>Ex:</u> $M = \begin{pmatrix} 0.75 & 0.02 & 0.10 \\ 0.20 & 0.90 & 0.20 \\ 0.05 & 0.08 & 0.70 \end{pmatrix}$ $\qquad D = \begin{pmatrix} 1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$

<u>$\lambda = 1$:</u> $\begin{pmatrix} -0.25 & 0.02 & 0.10 \\ 0.20 & -0.10 & 0.20 \\ 0.05 & 0.08 & -0.30 \end{pmatrix} \to \begin{pmatrix} 5 & 8 & -30 \\ -25 & 2 & 10 \\ 20 & -10 & 20 \end{pmatrix} \to \begin{pmatrix} 5 & 8 & -30 \\ 0 & 42 & -140 \\ 0 & -42 & 140 \end{pmatrix}$

$\to \begin{pmatrix} 5 & 8 & -30 \\ 0 & 42 & -140 \\ 0 & 0 & 0 \end{pmatrix}$

$5x + 8y - 30z = 0$

$42y - 140z = 0$

z free

$y = \frac{140z}{42} = \frac{10}{3}z$

$5x = 30 \cdot z - 8 \cdot \frac{10}{3}z = \frac{90 - 80}{3}z$

$x = \frac{2}{3}z$

$\frac{2}{3}z + \frac{10}{3}z + z = 1$

$5z = 1$

$z = 1/5$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2/3 \cdot z \\ 10/3 \cdot z \\ z \end{pmatrix} = z/3 \cdot \begin{pmatrix} 2 \\ 10 \\ 3 \end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix} 2/15 \\ 10/15 \\ 3/15 \end{pmatrix}$ (with $z = 1/5$)

<u>Conclusion:</u> As $n \to \infty$ (in the long run) $u = 2/15 \approx 13.3\%$ of families are urban, $s = 10/15 \approx 66.7\%$ are suburban, and $r = 3/15 = 20\%$ are rural.

<u>Check:</u> Compute M^{10}, M^{50}, M^{100} using Wolfram Alpha or other software.

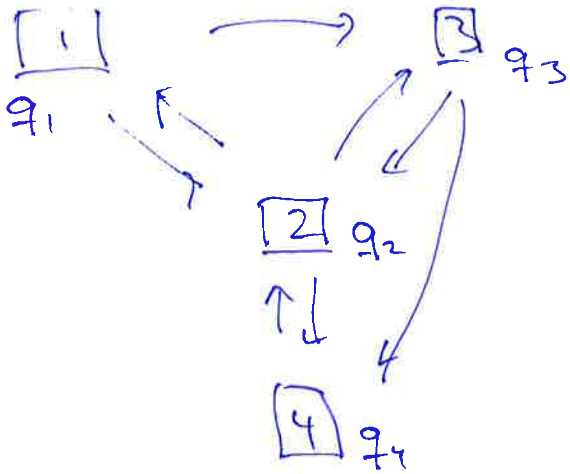
It is also possible to compute M^n as

$$M^n = P \cdot \begin{pmatrix} \overset{=1}{\lambda_1} & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}^n \cdot P^{-1} \approx P \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} P^{-1}$$

since $\lambda_1 = 1, \lambda_2, \lambda_3 < 1$

Ex: Google's Pagerank algorithm

Webpages $\boxed{1}$ – $\boxed{4}$ with links: q_i is the ranking no.



$$q_1 = K \cdot (q_2)$$
$$q_2 = K \cdot (q_1 + q_3 + q_4)$$
$$q_3 = K \cdot (q_1 + q_2)$$
$$q_4 = K \cdot (q_2 + q_3)$$

$$\underline{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}$$

$$\underline{q} = K \cdot \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & \cancel{0} 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \underline{q}$$

$$\lambda \cdot \underline{q} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \underline{q}$$

replace with M

$$M = \begin{pmatrix} 0 & 1/3 & 0 & 0 \\ 1/2 & 0 & 1/2 & 1 \\ 1/2 & 1/3 & 0 & 0 \\ 0 & 1/3 & 1/2 & 0 \end{pmatrix}$$

Eigenvectors for M with $\lambda = 1$,

$$\underline{V} = \begin{pmatrix} 4/29 \\ 12/29 \\ 6/29 \\ 7/29 \end{pmatrix} \qquad \begin{aligned} q_1 &= 4/29 \\ q_2 &= 12/29 \\ q_3 &= 6/29 \\ q_4 &= 7/29 \end{aligned}$$

$$M^n \longrightarrow (\underline{V} \,|\, \underline{V} \,|\, .. \,|\, \underline{V}) \qquad \text{efficient computation}$$

as $n \to \infty$

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2 Quadratic forms

Defn: A quadratic form in n variables $x_1, \dots, x_n$ is a polynomial function where each term has degree two.

Ex:

$n=1$: $f(x) = ax^2 \qquad (x = x_1)$

$n=2$: $f(x,y) = ax^2 + bxy + cy^2 \qquad (x = x_1, y = x_2)$

$n=3$: ~~$f(x,y,z)$~~

$$f(x_1,x_2,x_3) = c_{11}x_1^2 + c_{12}x_1x_2 + c_{13}x_1x_3 + c_{22}x_2^2 + c_{23}x_2x_3 + c_{33}x_3^2$$

general n: $f(x_1, \dots, x_n) =$

$$c_{11}x_1^2 + c_{12}x_1x_2 + \dots + c_{1n}x_1x_n + c_{22}x_2^2 + \dots + c_{2n}x_2x_n + \dots + c_{nn}x_n^2$$

Matrix form: $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

$\underline{x}^T \cdot A \cdot \underline{x}$: Ex: $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ gives:

$$(x_1 \; x_2) \cdot \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 + 2x_2 & 2x_1 + x_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= x_1^2 + 2x_2x_1 + 2x_1x_2 + x_2^2$$

$$= x_1^2 + 4x_1x_2 + x_2^2$$

$a_{ii} \rightsquigarrow a_{ii}x_i^2$

$a_{ij}, a_{ji} \rightsquigarrow a_{ij}x_ix_j \quad (i<j)$

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Note: Any quadratic form $f(x_1, \dots, x_n)$ can be written in matrix form

$$f(\underline{x}) = \underline{x}^T A \underline{x}$$

There is a unique matrix form with A symmetric.

Ex:

$$f(x_1, x_2, x_3) = \underbrace{x_1^2 + 7x_1x_2 - x_1x_3 + x_2^2 + 4x_2x_3 - x_3^2}_{\text{quadratic form}}$$

$$A = \begin{pmatrix} 1 & 7/2 & -1/2 \\ 7/2 & 1 & 2 \\ -1/2 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 7/2 & -1/2 \\ 7/2 & 1 & 2 \\ -1/2 & 2 & -1 \end{pmatrix}$$

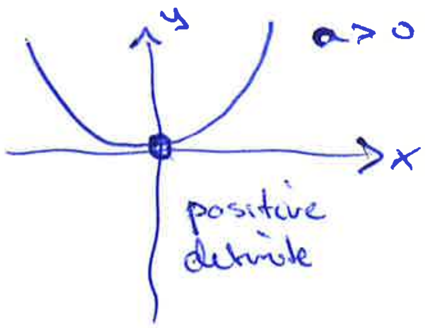
Ex:

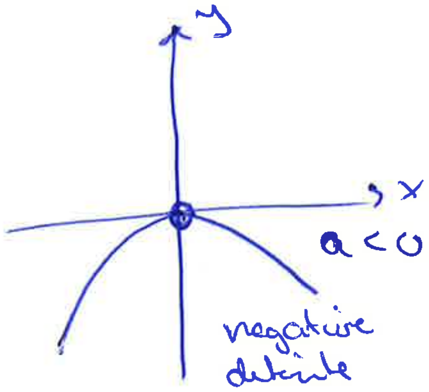
$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & -1 & 4 \\ 0 & 4 & 3 \end{pmatrix}$$

Symmetric matrix

$$f(x_1, x_2, x_3) = x_1^2 + 4x_1x_2 - x_2^2 + 8x_2x_3 + 3x_3^2$$

Ex: $n=1$: $f(x) = ax^2$





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Defn: $f(\underline{x})$ quadratic form in n variables
A symmetric $n \times n$-matrix

A/f is positive definite if $f(\underline{x}) > 0$ for all $\underline{x} \neq \underline{0}$
" positive semidefinite " $f(\underline{x}) \geq 0$ for all $\underline{x}$
" negative definite " $f(\underline{x}) < 0$ for all $\underline{x} \neq \underline{0}$
" negative semidefinite " $f(\underline{x}) \leq 0$ for all $\underline{x}$
" indefinite " $f(\underline{x})$ takes both positive and negative values
(i.e. none of the above holds)

$A = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

Ex: $f(x,y) = x^2 + 4y^2$
pos. defn.
$f(x,y) > 0$ for all $(x,y) \neq (0,0)$

Ex: $f(x,y) = x^2 + 4xy + 3y^2$

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$$

Result:
$f(\underline{x})$ quadr. form with symm. $n \times n$-matrix A
If A has eigenvalues $\lambda_1, \dots, \lambda_n$, then:

$\lambda_1, \dots, \lambda_n > 0 \iff$ positive definite
$\lambda_1, \dots, \lambda_n \geq 0 \iff$ positive semidefinite
$\lambda_1, \dots, \lambda_n < 0 \iff$ negative definite
$\lambda_1, \dots, \lambda_n \leq 0 \iff$ negative semidefinite
Both positive and negative eigenvalues $\iff$ indefinite

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

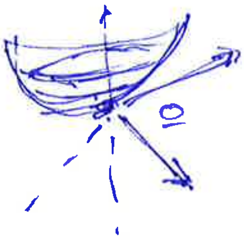
$$\lambda^2 - 4\lambda - 1 = 0$$

$$\lambda = \frac{4 \pm \sqrt{16+4}}{2} = 2 \pm \sqrt{5}$$

$\lambda_1 = 2 + \sqrt{5} > 0$
$\lambda_2 = 2 - \sqrt{5} < 0$
indefinite

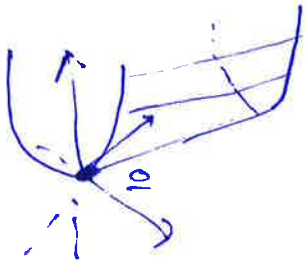
Interpretation: Graph of f

f pos. defn.:



$\underline{x} = \underline{0}$ global min.

f pos. semidefn.:

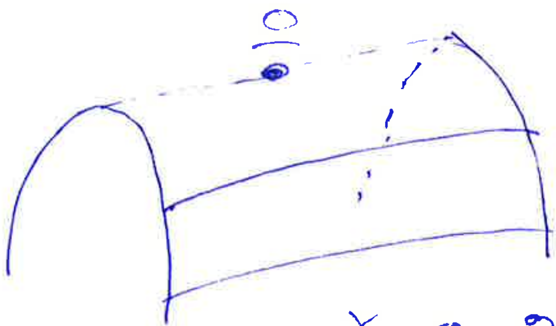


$\underline{x} = \underline{0}$ global min (and some other pts)

f neg. defn

$\underline{x} = \underline{0}$ global max.

f neg. semidefn.



$\underline{x} = \underline{0}$ global max

(and some other pts).

f indefinite

$\underline{x} = \underline{0}$ saddle pt.

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Ex: $f(\underline{x}) = 3x_1^2 + 2x_1x_3 - x_2^2 + x_3^2$ $\qquad A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

can compute eigenvalues

Principal minors:

Leading principal minors:

D_i : leading principal minor of order i

$D_i = M_{1,2,\dots,i,\,12\dots i}$ (first i rows, " i cols)

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$D_1 = 3$
$D_2 = 3\cdot(-1) - 0^2 = -3$
$D_3 = |A| = 1\cdot 1 + 1\cdot(-3) = -2$

Result:

$D_1 > 0, D_2 > 0, \dots, D_n > 0 \iff$ positive definite

$D_1 < 0, D_2 > 0, D < 0, \dots \iff$ negative definite

Ex: $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{pmatrix}$

$D_1 = -1 = \lambda_1$
$D_2 = 2 = \lambda_1\lambda_2$
$D_3 = -8 = \lambda_1\lambda_2\lambda_3$

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Result:

pos. semidefinite $\iff \Delta_1, \Delta_2, \ldots, \Delta_n \geq 0$

neg. semidefinite $\iff \Delta_1 \leq 0, \Delta_2 \geq 0, \ldots$

indefinite $\iff$ all other cases

Δ_i: principal minor of order i

= minor of order i where i choose the same rows as columns

Note: Many Δ_i's for each order i, the first of them is D_i.

Ex: $A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & -1 \\ 4 & -1 & 2 \end{pmatrix}$

$D_1 = 1$

$D_2 = -1 \Leftarrow \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}$

$D_3 = |A| = 4 \cdot (-14) + 1(-9) + 2 \cdot (-1)$

$= -56 - 9 - 2 = -67$

Choose one row and the same col. $\rightarrow$ Δ_1: $\overset{D_1}{M_{11}} = 1$ $\quad M_{22} = 3$ $\quad M_{33} = 2$

two rows same two cols. $\rightarrow$ Δ_2: $\underset{D_2}{M_{12,12}} = -1$ $\quad M_{13,13} = \begin{vmatrix} 1 & 4 \\ 4 & 2 \end{vmatrix} = -14$

$M_{23,23} = \begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix} = 5$

all three rows and cols $\rightarrow$ Δ_3: $\overset{D_3}{M_{123,123}} = |A| = -67$

A is indefinite

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Ex: $f = x^2 + 2xy + y^2 + z^2$ $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\left.\begin{aligned} D_1 &= 1 \\ D_2 &= 0 \\ D_3 &= 0 \end{aligned}\right\}$$ can be positive semidefinite, but must compute all principal minors to tell (it may also be indefinite).

D_1

$\Delta_1 = 1, \; 1, \; 1$

D_2

$\Delta_2 = 0, \; 1, \; 1$

$\Delta_3 = 0$

D_3

choose: 1,1 2,2 3,3

choose: 12,12 , 13,13, 23,23

choose: 123,123

$\Delta_1, \Delta_2, \Delta_3 \geq 0 \Rightarrow$ positive semidefinite