LECTURE 3

GRA 6035 MATHEMATICS BI

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SEP 13, 2016

Plan:

1. Review: Solving linear systems using minors.
2. Vectors, span and linear independence
3. Rank and linear independence

Reading:

[ME] 10.1-10.3, (10.4-10.7), 11.1

1 Linear systems and minors

Ex:

$$\begin{aligned} x+y+z+w &= 4 \\ x-2y+3z-w &= 2 \\ x+3y-z &= 1 \end{aligned}$$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -2 & 3 & -1 \\ 1 & 3 & -1 & 0 \end{pmatrix}$$

coeff. matrix

$$M_{123,123} = 1\cdot(-7) - 1\cdot(-4) + 1\cdot 5 = \underline{2} \neq 0$$

$$(A|\underline{b}) = \hat{A} = \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 1 & -2 & 3 & -1 & 2 \\ 1 & 3 & -1 & 0 & 1 \end{array}\right)$$

augmented matrix

$\text{rk}(A) = \underline{\underline{3}}$

$$\boxed{\begin{aligned} x+y+z &= 4-w \\ x-2y+3z &= 2+w \\ x+3y-z &= 1 \end{aligned}}$$

The linear system has one free variable (w), and infinitely many solutions.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 3 \\ 1 & 3 & -1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 4-w \\ 2+w \\ 1 \end{pmatrix} \Leftarrow \begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 3 \\ 1 & 3 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4-w \\ 2+w \\ 1 \end{pmatrix}$$

Conclusion:

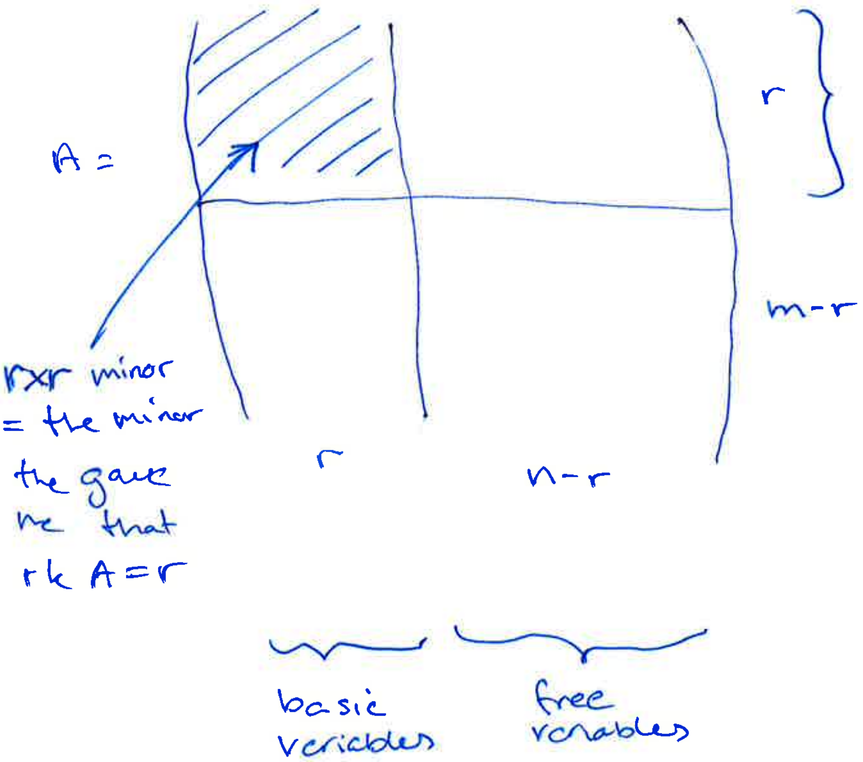
$$\left.\begin{array}{l} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = b_m \end{array}\right\} \iff \underline{A}\underline{x} = \underline{b}$$

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

$$\hat{A} = (A|\underline{b}) = \left(A \,\middle|\, \underline{b} \right)$$

i) If $\text{rk}\, A < \text{rk}\, \hat{A}$, then the system is inconsistent (no sol'ns)

ii) If $\text{rk}\, A = \text{rk}\, \hat{A}$, then the system is consistent (has sol'ns with $n - \text{rk}(A)$ degrees of freedom



Assume $\text{rk}\, A = r = \text{rk}\, \hat{A}$.

Use the rows

Solve the basic variables in terms of the free.

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2 Vectors

Defn: An m-vector (column vector) is an $m \times 1$-matrix

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$$

Ex: $\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ 3-vector $\quad \underline{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ 2-vector

Computing with vectors:

Addition s, subtraction, scalar multiplication: As for matrices

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \qquad 3 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

Linear combinations of vectors:

If $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n$ are m-vectors, then a linear combination is an expression

$$c_1 \cdot \underline{v}_1 + c_2 \cdot \underline{v}_2 + \ldots + c_n \cdot \underline{v}_n$$

where $c_1, c_2, \ldots, c_n$ are numbers.

Defn: $\text{span}(\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n) = \{ c_1\underline{v}_1 + \ldots + c_n\underline{v}_n : c_1, \ldots, c_n \in \mathbb{R} \}$

= all linear combinations (all choices of $c_1, \ldots, c_n$)

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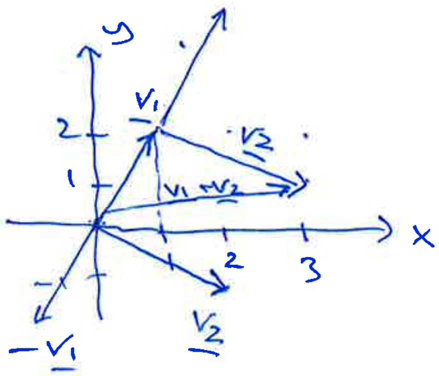
Ex: $\underline{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\quad \underline{v}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$



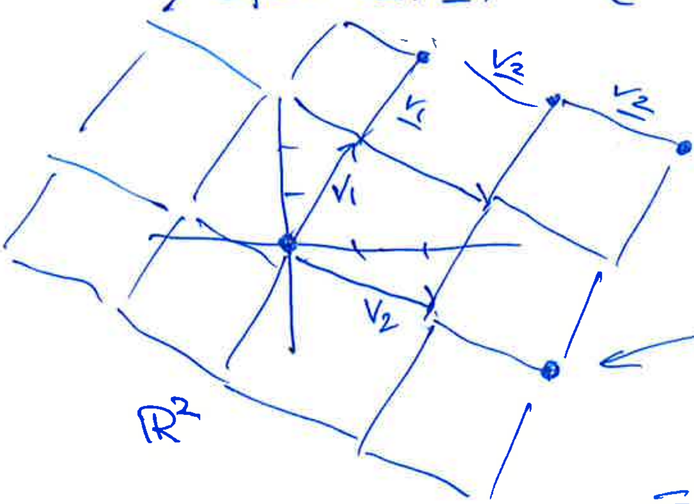
Think of vectors as displacements

$\underline{v}_1 + \underline{v}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$2\underline{v}_1 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

$-1 \cdot \underline{v}_1 = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$

$\text{span}(\underline{v}_1, \underline{v}_2) = \left\{ c_1 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} : c_1, c_2 \in \mathbb{R} \right\}$



$2 \cdot \underline{v}_1 + 2 \cdot \underline{v}_2$

$0 \cdot v_1 + 2 \cdot \underline{v}_2$

$= \text{span}(\underline{v}_1, \underline{v}_2) =$ all vectors in the plane.

$\mathbb{R}^n$: Euclidean n-space, all n-vectors

$\mathbb{R}$: real numbers

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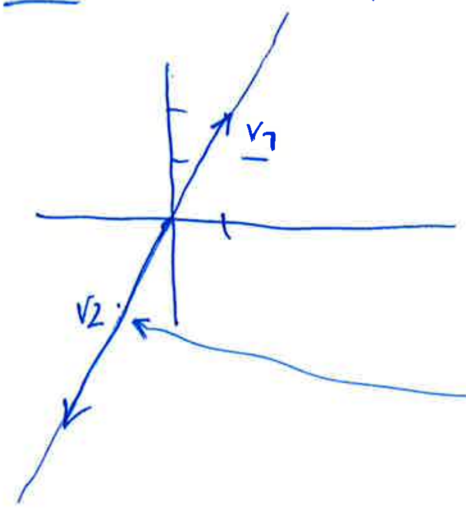
Ex: $\quad v_1 = \begin{pmatrix}1\\2\end{pmatrix} \quad \underline{v_2} = \begin{pmatrix}-2\\-4\end{pmatrix}$



span$(\underline{v_1}, v_2)$ = { all linear comb. $c_1 \cdot \underline{v_1} + c_2 \cdot \underline{v_2}$ }

= the line

Ex: Span$(\underline{v_1}, \underline{v_2})$ where $\underline{v_1} = \begin{pmatrix}1\\2\end{pmatrix}$, $\underline{v_2} = \begin{pmatrix}3\\-1\end{pmatrix}$

Is $\begin{pmatrix}5\\2\end{pmatrix}$ in the span?

$$\begin{pmatrix}5\\2\end{pmatrix} = x_1 \cdot \begin{pmatrix}1\\2\end{pmatrix} + x_2 \cdot \begin{pmatrix}3\\-1\end{pmatrix}$$ ← vector equation

$$\begin{pmatrix}5\\2\end{pmatrix} = \begin{pmatrix}x_1\\2x_1\end{pmatrix} + \begin{pmatrix}3x_2\\-x_2\end{pmatrix} = \begin{pmatrix}x_1+3x_2\\2x_1-x_2\end{pmatrix}$$

$$\begin{aligned} x_1 + 3x_2 &= 5 \\ 2x_1 - x_2 &= 2 \end{aligned}$$

$$\left(\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & -1 & 2 \end{array}\right) \; -2$$

$$\left(\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & -7 & -8 \end{array}\right)$$

$$\begin{aligned} x_1 + 3x_2 &= 5 \\ -7x_2 &= -8 \end{aligned}$$

$$\begin{cases} x_1 = 5 - 3 \cdot 8/7 = \underline{11/7} \\ x_2 = \underline{8/7} \end{cases}$$

$$\begin{pmatrix}5\\2\end{pmatrix} = \frac{11}{7} \cdot \underline{v_1} + \frac{8}{7} \cdot \underline{v_2}$$

<u>It is in the span.</u>

Since $x_1 \cdot \begin{pmatrix}1\\2\end{pmatrix} + x_2 \cdot \begin{pmatrix}3\\-1\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix}$

has one unique solution for all a, b,

Span$(\underline{v_1}, \underline{v_2}) = \mathbb{R}^2$

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Ex: $\underline{v_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\underline{v_2} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\underline{v_3} = \begin{pmatrix} 1 \\ 4 \\ 9 \end{pmatrix}$

Find span$(\underline{v_1}, \underline{v_2}, \underline{v_3})$:

$$x_1 \cdot \underline{v_1} + x_2 \cdot \underline{v_2} + x_3 \cdot \underline{v_3} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 1 & 2 & 4 & b \\ 1 & 3 & 9 & c \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 3 & * \\ 0 & 2 & 8 & * \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 3 & * \\ 0 & 0 & 2 & * \end{array}\right)$$

augmented.

one solution

Conclusion:

$$\text{span}(\underline{v_1}, \underline{v_2}, \underline{v_3}) = \mathbb{R}^3$$

Alt: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$ $A \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$|A| = 2 \neq 0 \implies \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Ex: $\underline{v_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\underline{v_2} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ $\underline{v_3} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$

span$(\underline{v_1}, \underline{v_2}, \underline{v_3}) = ?$ $A \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 0 & -2 \end{pmatrix} \qquad |A| = 1 \cdot 2 + (-2) \cdot 1 = 0$$

$$\hat{A} = \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 1 & 2 & 4 & b \\ 1 & 0 & -2 & c \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 3 & b-a \\ 0 & -1 & -3 & c-a \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 3 & b-a \\ 0 & 0 & 0 & b+c-2a \end{array}\right)$$

$b+c-2a=0$: inf. many solutions

$b+c-2a \neq 0$: no solutions.

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$$\text{Span}(\underline{v_1}, \underline{v_2}, \underline{v_3}) = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} : \quad b + c = 2a \right\}$$

2-dimensional

(3) Linear independence

Defn: The m-vectors $\underline{v_1}, \underline{v_2}, \ldots, \underline{v_n}$ are called <u>linearly dependent</u> if at least one of the vectors can be written as a linear combination of the others

and

<u>linearly independent</u> if this is not the case

Ex: $\underline{v_3} = \underline{v_1} + 3\underline{v_2}$ $\qquad \{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$ linearly dependent

$\underline{0} = \underline{v_1} + 3\underline{v_2} - \underline{v_3}$

$\Downarrow$

$\text{span}(\underline{v_1}, \underline{v_2}, \underline{v_3}) = \text{span}(\underline{v_1}, \underline{v_2})$:

$$\text{span}(\underline{v_1}, \underline{v_2}, \underline{v_3}) = \{ c_1 \cdot \underline{v_1} + c_2 \underline{v_2} + c_3 \underline{v_3} \}$$

$$= \{ c_1 \underline{v_1} + c_2 \underline{v_2} + c_3 \cdot (\underline{v_1} + 3\underline{v_2}) \}$$

$$= \{ (c_1 + c_3)\underline{v_1} + (c_2 + 3c_3)\underline{v_2} \}$$

$\underline{v_1} = -3\underline{v_2} + \underline{v_3}$

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Useful formulation:

$\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$ **linearly dependent** $\iff$ $x_1\underline{v}_1 + x_2\underline{v}_2 + \dots + x_n\underline{v}_n = \underline{0}$ has a non-trivial solution, i.e. a solution $(x_1, x_2, \dots, x_n) \neq (0, 0, \dots, 0)$

linearly independent $\iff$ $x_1\underline{v}_1 + x_2\underline{v}_2 + \dots + x_n\underline{v}_n = \underline{0}$ has only the trivial solution $(x_1 \dots x_n) = (0, \dots, 0)$

Ex: $\underline{v}_3 = \underline{v}_1 + 3\underline{v}_2$

$\underline{0} = \underline{v}_1 + 3\underline{v}_2 - \underline{v}_3$

$(1, 3, -1)$ is a solution, non-trivial.

Ex: $\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\underline{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ $\underline{v}_3 = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$

$$x_1 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + x_2 \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

homogeneous linear system

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 0 & -2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 4 & 0 \\ 1 & 0 & -2 & 0 \end{array}\right) \begin{matrix} \\ -1 \\ -1 \end{matrix}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & -1 & -3 & 0 \end{array}\right) \begin{matrix} \\ \\ 1 \end{matrix} \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 = 2x_3$
$x_2 = -3x_3$
x_3 free

$\underline{x_3 = 1}$: $x_1 = 2$
$x_2 = -3$

1 degree of freedom $\Rightarrow$ inf. many solutions $\Rightarrow$ many non-trivial solutions

$2\underline{v}_1 - 3\underline{v}_2 + \underline{v}_3 = \underline{0}$

$\underline{v}_3 = -2\underline{v}_1 + 3\underline{v}_2$

Concl: $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ are linearly **dependent**

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$\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \underline{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \quad v_3 = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$

<u>Found:</u> $\underline{v}_3 = -2\underline{v}_1 + 3\underline{v}_2 \Rightarrow \text{span}(\underline{v}_1, \underline{v}_2, v_3)$

$= \text{span}(\underline{v}_1, v_2)$

$= \left\{ c_1 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right\}$

<u>In general:</u> How to determine linear dependence / independence

$\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n\} \longrightarrow A = (\underline{v}_1 | \underline{v}_2 | \ldots | \underline{v}_n) \rightsquigarrow$ find pivot positions in A

coeff. matrix of linear system

$x_1 \underline{v}_1 + x_2 \underline{v}_2 \ldots + x_n \underline{v}_n = \underline{0}$

<u>Conclusion:</u>

If A has n pivot positions, then $\{\underline{v}_1, \ldots, \underline{v}_n\}$ <u>linearly independent</u>

If A has less than n pivot positions, then $\{\underline{v}_1, \ldots, \underline{v}_n\}$ are <u>linearly dependent</u>

Pick the vectors $\underline{v}_i$ that correspond to pivot positions. This always gives a maximal set of linearly independent vectors among $\{\underline{v}_1, \ldots, \underline{v}_n\}$

$$\boxed{\text{Rk}(A) = \#\text{ pivot positions in } A = \text{maximal number of linearly independent vectors among } \{\underline{v}_1, \ldots, \underline{v}_n\}}$$

Proof:

$\{\underline{v}_1, \ldots, \underline{v}_n\}$ vectors $\longrightarrow$ $\boxed{\begin{array}{l} A = (\underline{v}_1 | \underline{v}_2 | \ldots | \underline{v}_n) \\ \text{Lin.sys: } A \cdot \underline{x} = \underline{0} \end{array}}$

rk $A = n$: $n - \text{rk}A = n - n = \underline{0}$ degrees of freedom $\Rightarrow$ unique sol'n $\underline{x} = \underline{0}$
linearly independent

rk $A < n$: $n - \text{rk}A > 0$ free variables $\Rightarrow$ infinitely many sol'ns,
linearly dependent

If we take away some vectors from $\underline{v}_1, \ldots, \underline{v}_n$, it corresponds to deleting the corresponding col's in A. If we remove all non-pivot col's, then there will be no free var's left, and the new system becomes one with unique sol'n $\underline{x} = \underline{0}$, i.e. the remaining vectors ($\Leftrightarrow$ pivot col's) are linearly independent.

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Special case: $m = n$

$\{\underline{v}_1, \ldots, \underline{v}_n\} \rightsquigarrow A = (\underline{v}_1 | \underline{v}_2 | \ldots | \underline{v}_n) \rightsquigarrow$ Compute $|A|$.

$|A| \neq 0$: $\{\underline{v}_1, \ldots, \underline{v}_n\}$ linearly independent

$|A| = 0$: — " — linearly dependent

Ex: $\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\underline{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ $\underline{v}_3 = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} \rightsquigarrow A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 0 & -2 \end{pmatrix}$

$|A| = 1 \cdot 2 + (-2) \cdot 1 = 0 \Rightarrow \{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ Lin. independent.