LECTURE 2

GRA6035 MATHEMATICS

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Plan:

1. Matrices and matrix algebra
2. Determinants
3. Minors, rank and linear algebra

Reading:

[ME] 8.1-8.4, (8.5-8.6), 9.1, 9.2, (9.3), 26.1-26.3, (26.4), 26.5

FORK1003: Math, Lecture 2-3

① Matrices and matrix algebra

Defn: An $m\times n$-matrix A is a rectangular block (m rows, n col's) of numbers.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij})$$

a_{ij}: coeff. in A in position (i,j), i.e. row i col. j

Ex: $A = \begin{pmatrix} 2 & 4 & 7 \\ 3 & -1 & 2 \end{pmatrix}$ 2×3-matrix $B = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}$ 2×2-matrix

Operations on matrices:

* **Addition/subtraction:** position by position, defined if the matrices have the same size

Ex: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ 7 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 10 & 2 \end{pmatrix}$

* **Matrix multiplication:** $A\cdot B$ is defined if $\#\text{cols}(A) = \#\text{row}(B)$

Ex: $\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} 7 & 1 & 0 \\ 4 & 0 & -1 \end{pmatrix} = \begin{pmatrix} \cdot & \underset{\uparrow}{1} & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} = \begin{pmatrix} 15 & 1 & -2 \\ 10 & 2 & 1 \end{pmatrix}$

$2\times 2 = 2\times 3$ $1\cdot 1 + 2\cdot 0$

Note: $A \cdot B \neq B \cdot A$!

Ex: $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$\neq$

$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

* scaler multiplication : multiplication of a number (scalar) and a matrix, computed position by position

Ex: $2 \cdot \begin{pmatrix} 1 & 3 & 0 \\ 4 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 6 & 0 \\ 8 & -2 & 2 \end{pmatrix}$

$\begin{pmatrix} 1 & 3 & 0 \\ 4 & -1 & 1 \end{pmatrix} \cdot 2 = \begin{pmatrix} 2 & 6 & 0 \\ 8 & -2 & 2 \end{pmatrix}$

Properties:

Ex: $(A+B)\cdot(A-B) = A \cdot A + B \cdot A + A \cdot (-B) + B \cdot (-B)$

$= \underline{A^2 + BA - AB - B^2}$

Special matrices:

$0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ zero matrix

$A + 0 = A$
$0 + A = A$
for all matrices A

$I = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ identity matrix

$A \cdot I = A$
$I \cdot A = A$
for all matrices A

$I = I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\vdots$

Rules for matrix operations

1) $A+B = B+A$
2) $A\cdot(B+C) = AB+AC$
3) $(A+B)+C = A+(B+C)$
4) $(AB)C = A(BC)$

$\vdots$

But $\underline{AB \neq BA}$.

Transpose:

A $m\times n$-matrix $\rightsquigarrow$ $A^T = A^t$ $n\times m$-matrix, the transpose of A

Ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ $\qquad B = \begin{pmatrix} 4 & 1 & 3 \\ 1 & -1 & 0 \\ 3 & 0 & 2 \end{pmatrix}$

$\Downarrow$ $\qquad\qquad\qquad$ $\Downarrow$

$A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$ $\qquad B^T = \begin{pmatrix} 4 & 1 & 3 \\ 1 & -1 & 0 \\ 3 & 0 & 2 \end{pmatrix}$

An $n\times n$-matrix is called symmetric if $A^T = A$.

A symmetric $\iff$ $\begin{cases} A \text{ is square} & (\#\text{rows} = \#\text{cols}) \\ \text{and} & \\ a_{ij} = a_{ji} \text{ for all } i,j. \end{cases}$

Properties: i) $(A+B)^T = A^T + B^T$

ii) $(c \cdot A)^T = c \cdot A^T$ (c number)

iii) $(A \cdot B)^T = B^T \cdot A^T$

Powers: If A is square $(n \times n)$, then we can define

$$A^2 = A \cdot A$$

$$A^3 = A \cdot A \cdot A = (A \cdot A) \cdot A = A \cdot (A \cdot A)$$

$$A^4 = A \cdot A \cdot A \cdot A = A^2 \cdot A^2$$

$$\vdots$$

$$A^n$$

Ex: Compute $\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}^n = A^n$

$$A^2 = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}^2 = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 8 \\ -4 & 7 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}^4 = \underbrace{\begin{pmatrix} -1 & 8 \\ -4 & 7 \end{pmatrix}}_{\shortparallel \atop A^2} \cdot \underbrace{\begin{pmatrix} -1 & 8 \\ -4 & 7 \end{pmatrix}}_{\shortparallel \atop A^2} = \begin{pmatrix} -31 & 48 \\ -24 & 17 \end{pmatrix}$$

$$\vdots$$

Difficult to compute A^n!

Inverse matrices:

A $n\times n$-matrix : The inverse of A is a matrix B $(n\times n)$ s.t.

$$A\cdot B = I_n$$
$$B\cdot A = I_n$$

Fact:

- if the inverse exists, it is unique and it is written $A^{-1} = B$
- the inverse A^{-1} exists if and only if $|A| \neq 0$.

Ex: $n=2$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

If $\underbrace{ad-bc}_{|A|}=0$: there is no inverse

If $ad-bc\neq 0$: $A^{-1} = \frac{1}{ad-bc}\cdot\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Ex: $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{-3}\cdot\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{pmatrix}$

$$A\cdot A^{-1} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\cdot\begin{pmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = A^{-1}\cdot A$$

Ex: $x + 2y = 5$
$2x + y = 17$

$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ $(A|\underline{b}) = \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & 1 & 17 \end{array}\right)$

$\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ $\underline{b} = \begin{pmatrix} 5 \\ 17 \end{pmatrix}$

$$\boxed{A\cdot\underline{x} = \underline{b}}$$
matrix form of the linear system

$\xrightarrow{\text{if } A^{-1} \text{ exists}}$ $\cancel{A^{-1}}\cdot\cancel{A}\underline{x} = A^{-1}\cdot\underline{b}$

$\underline{x} = A^{-1}\cdot\underline{b} = \begin{pmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{pmatrix}\cdot\begin{pmatrix} 5 \\ 17 \end{pmatrix}$

$= \ldots$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1\cdot x + 2y \\ 2x + 1\cdot y \end{pmatrix} = \begin{pmatrix} x+2y \\ 2x+y \end{pmatrix} = \begin{pmatrix} 5 \\ 17 \end{pmatrix}$$

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(2) Determinants

$A \rightsquigarrow \det(A) = |A|$

$n \times n$-matrix

determinant of A, a number

Property:

$|A| \neq 0 \implies A^{-1}$ exists

$|A| = 0 \qquad A^{-1}$ does not exist

How to define and compute $|A|$:

$n=1$: $A = (a)$ $\qquad |A| = a$

$n=2$: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\qquad |A| = ad - bc$

$n=3$: $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{matrix} a & b \\ d & e \\ g & h \end{matrix}$ $\qquad |A| = aei + bfg + cdh - ceg - afh - bdi$

$A = \begin{pmatrix} a & b & c \\ d & \boxed{e} & \boxed{f} \\ g & \boxed{h} & \boxed{i} \end{pmatrix}$ $\qquad = a(ei - fh) - b(di - fg) + c(dh - eg)$

cofactor expansion

Method of cofactor expansion:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Cofactor expansion along the first row:

$$|A| = a_{11} \cdot C_{11} + a_{12} \cdot C_{12} + a_{13} \cdot C_{13} + \dots + a_{1n} \cdot C_{1n}$$

$$\underbrace{C_{ij}}_{\substack{\text{cofactor} \\ (i,j)}} = \underbrace{(-1)^{i+j}}_{\text{sign}} \cdot \underbrace{M_{ij}}_{\substack{\text{minor} \\ (i,j)}} ,$$

M_{ij} = determinant of the submatrix you get by deleting row i and column j from A.

Ex:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} = +1 \cdot \begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= 1 \cdot (18-12) - 1(9-4) + 1 \cdot (3-2)$$

$$= 6 - 5 + 1 = \underline{\underline{2}}$$

sign : $(-1)^{i+j}$

$$\begin{pmatrix} + & - & + & - \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

Facts:

* Cofactor expansion can be used to compute any determinant.
* Cofactor expansion along any row or column gives $\det(A)$.

Properties:

i) $|AB| = |A| \cdot |B|$

ii) $|A^T| = |A|$

iii) $|A^{-1}| = 1/|A|$

iv) $|c \cdot A| = c^n \cdot |A|$ if c is a number, A is $n \times n$-matrix

Efficient computation of determinants

Ex:

$$A = \begin{pmatrix} 2 & 7 & -1 & 14 \\ 0 & 1 & 3 & \sqrt{41} \\ 0 & 0 & 17 & 1 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

upper triangular matrix $\Rightarrow$ $|A| = a_{11} \cdot a_{22} \cdots a_{nn}$

($a_{ij} = 0$ when $i > j$)

product of coeff. on the diagonal.

$$|A| = +2 \cdot \begin{vmatrix} 1 & 3 & \sqrt{41} \\ 0 & 17 & 1 \\ 0 & 0 & 4 \end{vmatrix} - 0 \cdot * + 0 \cdot * - 0 \cdot *$$

$$= 2 \cdot \left(1 \cdot \begin{vmatrix} 17 & 1 \\ 0 & 4 \end{vmatrix} - 0 \cdot * + 0 \cdot * \right)$$

$$= 2 \cdot 1 \cdot 17 \cdot 4 = \underline{136}$$

Fact: $A \to \to \cdots \to E$ (echelon form) $|E|$ is easy to compute

i) $A \to B$ is interchanging two rows: $|B| = -|A|$

ii) " multiplying a row with $c \neq 0$: $|B| = c \cdot |A|$

iii) " add a multiple of one row to another row: $|B| = |A|$.

Ex:

$$\begin{vmatrix} 1 & 2 & 3 & 7 \\ 0 & 1 & -1 & 2 \\ -1 & 2 & 3 & 4 \\ 2 & 1 & 1 & 0 \end{vmatrix} \begin{matrix} \\ \\ \leftarrow 1 \\ \leftarrow -2 \end{matrix} = \begin{vmatrix} 1 & 2 & 3 & 7 \\ 0 & 1 & -1 & 2 \\ 0 & 4 & 6 & 11 \\ 0 & -3 & -5 & -14 \end{vmatrix} \begin{matrix} \\ -4 \\ \leftarrow \\ \leftarrow 3 \end{matrix} 1 \cdot \begin{vmatrix} 1 & -1 & 2 \\ 4 & 6 & 11 \\ -3 & -5 & -14 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 3 & 7 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 10 & 3 \\ 0 & 0 & -8 & -8 \end{vmatrix} = 1 \cdot (1 \cdot (-80 + 24)) = \underline{-56}$$

Ex: Compute

$$\begin{vmatrix} 4 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 6 & -2 & 0 \\ 1 & -1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{vmatrix}.$$

Sol:

$$\begin{vmatrix} 4 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 6 & -2 & 0 \\ 1 & -1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{vmatrix} \quad -\frac{1}{4} \quad -1/4$$

$$= \begin{vmatrix} 4 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 6 & -2 & 0 \\ 0 & -1 & 2 & 1/4 & 1/4 \\ 0 & 1 & 0 & 1/4 & 1/4 \end{vmatrix}$$

$$= 4 \begin{vmatrix} 2 & 0 & 1 & -1 \\ 0 & 6 & -2 & 0 \\ -1 & 2 & 1/4 & 1/4 \\ 1 & 0 & 1/4 & 1/4 \end{vmatrix} \quad -2 \quad 1$$

$$= 4 \cdot \begin{vmatrix} 0 & 0 & 1/2 & -3/2 \\ 0 & 6 & -2 & 0 \\ 0 & 2 & 1/2 & 1/2 \\ 1 & 0 & 1/4 & 1/4 \end{vmatrix}$$

$$= 4 \cdot 1 \cdot (-1) \cdot \begin{vmatrix} 0 & 1/2 & -3/2 \\ 6 & -2 & 0 \\ 2 & 1/2 & 1/2 \end{vmatrix}$$

$$= -4 \cdot \left(-6 \cdot \left(1/4 + 3/4 \right) + 2(-3) \right)$$

$$= -4 \cdot (-6 - 6) = -4 \cdot (-12) = \underline{\underline{48}}$$

③ Minors and rank

A

$m \times n$ matrix

Minor of rank r:

Determinant of an $r \times r$ submatrix of A.

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 7 & -1 & 3 \end{pmatrix}$$

Minors of order 2 = maximal minors

all 2-minors of A

$$\begin{cases} M_{12,12} = \begin{vmatrix} 1 & 2 \\ 7 & -1 \end{vmatrix} = \underline{-15} \\ M_{12,13} = \begin{vmatrix} 1 & 4 \\ 7 & 3 \end{vmatrix} = \underline{-25} \\ M_{12,23} = \begin{vmatrix} 2 & 4 \\ -1 & 3 \end{vmatrix} = \underline{10} \end{cases}$$

1-minors of A

$$\begin{cases} M_{1,1} = 1 & M_{21} = 7 \\ M_{1,2} = 2 & M_{22} = -1 \\ M_{13} = 4 & M_{23} = 3 \end{cases}$$

$$\boxed{\textbf{Fact: } \text{The rank of } A \text{ is equal to the maximal order of a non-zero minor in } A.}$$

Ex: $\operatorname{rk}\begin{pmatrix} 1 & 2 & 4 \\ 7 & -1 & 3 \end{pmatrix} = 2$ since $\begin{vmatrix} 1 & 2 \\ 7 & -1 \end{vmatrix} = -15 \neq 0$ is a non-zero 2-minor = maximal minor.

Linear systems:

Ex:

$$\begin{aligned} x+y+z+w &= 4 \\ x+2y+3z-w &= 2 \\ x+3y+z &= 1 \end{aligned}$$

$$A = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & -1 \\ 1 & 3 & 1 & 0 \end{array}\right)$$

$$M_{123,123} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 1 \end{vmatrix}$$

$$= 1\cdot(-7) - 1(-2) + 1\cdot 1$$

$$= -7+2+1 = \underline{-4} \neq 0$$

$$\begin{aligned} x+y+z &= 4-w \\ x+2y+3z &= 2+w \\ x+3y+z &= 1 \end{aligned}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 4-w \\ 2+w \\ 1 \end{pmatrix}$$

$\underline{\text{rk } A = 3}$

3 pivot positions in A

$\Rightarrow$ one free variable, inf. many solutions

w is free

Explanation:

Since $M_{123,123} = -4 \neq 0$, it means that the system can be written

$$\begin{aligned} x+y+z &= 4-w \\ x+2y+3z &= 2+w \\ x+3y+z &= 1 \end{aligned}$$

where we can solve for x, y, z in terms of w $\leftarrow$ this means w is free (can be chosen as free)

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4-w \\ 2+w \\ 1 \end{pmatrix}$$

$$\Updownarrow$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 4-w \\ 2+w \\ 1 \end{pmatrix} = \ldots$$

(same expressions in w)

$$\uparrow$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 1 \end{vmatrix} = M_{123,123} \neq 0$$