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LECTURE 11

GRA 6035

EIVIND ERIKSEN

NOV 10, 2016

MATHEMATICS

Plan:

1. Exact differential equations
2. Second order linear differential equations
3. Stability → Next lecture

Reading:

[MEJ] 24.1-24.3, (24.4-24.6)

Review:

First order differential equation $y' = F(y,t)$

1. Seperable $y' = f(y) \cdot g(t)$

$$\frac{1}{f(y)} \cdot y' = g(t) \quad \rightarrow \quad \int \frac{1}{f(y)}\, dy = \int g(t)\, dt$$

2. Linear $y' = b(t) - y \cdot a(t) \quad \leftrightarrow \quad y' + a(t)y = b(t)$

Integrating factor:

$$u(t) = e^{\int a(t)dt} \quad \rightarrow \quad (y \cdot u(t))' = b(t)u(t)$$

$$\rightarrow \quad y = \frac{1}{u(t)} \int b(t)u(t)\, dt$$

3. Exact

$$p(y,t) + q(y,t) \cdot y' = 0$$

s.t. $p = h'_t$ and $q = h'_y$ for some $h(y,t)$

$$\Updownarrow$$

$$\boxed{h(y,t) = C}$$

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1 Exact differential equations

A first order differential equation is called exact if it can be written as

$$h'_t + h'_y \cdot y' = 0$$

for some function $h = h(y,t)$. Then the solution is given by

$$\boxed{h(t,y) = C}$$

Ex: $1 + ty^2 + t^2y \cdot y' = 0$

$$t^2y \cdot y' = -(1+ty^2)$$

not separable
not linear $\longrightarrow y' = -\dfrac{1+ty^2}{t^2y} = F(y,t)$

Is it exact? $(1+ty^2) + (t^2y)y' = 0$

$p = 1+ty^2$
$q = t^2y$

$$\underset{\substack{\parallel \\ p}}{(1+ty^2)} + \underset{\substack{\parallel \\ q}}{(t^2y)} \cdot y' = 0$$

Equations for exactness:

$$\left.\begin{aligned} &(1)\quad 1+ty^2 = h'_t \\ &(2)\quad t^2y = h'_y \end{aligned}\right\} \text{ for some (unknown) function } h = h(y,t)$$

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(1) $1+ty^2 = h'_t$

(2) $t^2y = h'_y$

(1) $1+ty^2 = h'_t$

$$h = \int 1 + ty^2\, dt$$

$$= t + y^2 \cdot \tfrac{1}{2}t^2 + C(y)$$

Solution of (1): $h = t + \frac{1}{2}t^2y^2 + C(y)$

(2) $t^2y = h'_y$

$$t^2y = 0 + \cancel{\tfrac{1}{2}}t^2 \cdot \cancel{2}y + C'(y)$$

$$\underline{t^2y} = \underline{t^2y} + C'(y)$$

ok if $C'(y)=0$

$C(y) = K$

$= \underline{0}$

Solutions of (1) and (2):

$h = t + \frac{1}{2}t^2y^2 + K$ $\longrightarrow$ The differential eqn. is exact.

General solution:

$h = C$

$t+\frac{1}{2}t^2y^2+K = C$

$t+\frac{1}{2}t^2y^2 = C-K$

$t+\frac{1}{2}t^2y^2 = C$

$$\frac{\cancel{\frac{1}{2}t^2}y^2}{\cancel{\frac{1}{2}t^2}} = \frac{C-t}{\frac{1}{2}t^2}\frac{\cdot 2}{\cdot 2} \qquad y^2 = \frac{2(C-t)}{t^2} \qquad \underline{\underline{y = \pm\sqrt{\frac{2(C-t)}{t^2}}}}$$

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$$h = t + \frac{1}{2}t^2y^2 = C$$

implicit derivation

$y = y(t)$

$$\frac{d}{dt}\left(t + \frac{1}{2}t^2y^2\right) = \frac{d}{dt}(C)$$

$$1 + \frac{1}{2}\left(2t \cdot y^2 + t^2 \cdot 2y \cdot y'(t)\right) = 0$$

$$\frac{d}{dt}(y^2) = 2y \cdot y'$$

$$1 + ty^2 + t^2 y \cdot y' = 0$$

$$\frac{d}{dt}\left(h(y,t)\right) = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial y} \cdot \frac{dy}{dt} = h'_t + h'_y \cdot y'$$

$$\boxed{\text{Main point:} \quad h(y,t) = C \iff h'_t + h'_y \cdot y' = 0}$$

Notice:

$p + q \cdot y' = 0$ exact

$$\Updownarrow$$

$p = h'_t\,,\ q = h'_y$ for some $h = h(y,t)$

$$\Updownarrow$$

$$\frac{\partial p}{\partial y} = \frac{\partial q}{\partial t}$$

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Ex: $\underbrace{3t^2+y^2}_{p} + \underbrace{(2ty-2)}_{q} y' = 0 \quad , y(1)=2$

$t=1$
$y=2$

$h'_t + h'_y \cdot y' = 0 \longrightarrow$

$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial y} \cdot \frac{dy}{dt} = 0$

$$\underset{\partial h/\partial t}{\overset{\shortparallel}{h'_t}} = 3t^2+y^2 \quad , \quad \underset{\partial h/\partial y}{\overset{\shortparallel}{h'_y}} = 2ty-2$$

① $h'_t = 3t^2+y^2$

$$h = \int 3t^2+y^2 \, dt = \underline{t^3+y^2t+C(y)}$$

② $h'_y = 2ty-2$

$$\cancel{0} + \underline{2yt} + C'(y) = \underline{2ty}-2$$

ok if $C'(y) = -2$

$C(y) = \underline{-2y}$

$\Downarrow$

$$h = \underline{t^3} + \underline{ty^2} \underline{-2y} = C$$

$$ty^2 - 2y + (t^3-C) = 0$$

$a=t$
$b=-2$
$c=t^3-C$

$$y = \frac{2 \pm \sqrt{4-4t(t^3-C)}}{2t}$$

$t=1$
$y=2$

$$2 = \frac{2 \pm \sqrt{4-4(1-C)}}{2} = \frac{2 \pm \sqrt{4C}}{2} \quad | \cdot 2$$

$4 = 2 \pm \sqrt{4C} \qquad 2 = \pm\sqrt{4C}$

$C=1$
$+$

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$$y = \frac{2 + \sqrt{4 - 4t(t^3 - 1)}}{2t}$$

$$y = \frac{2 + \sqrt{4 + 4t - 4t^4}}{2t} = \frac{1 + \sqrt{1 + t - t^4}}{t}$$

② Linear second order differential equations

Defn: A linear second order differential equation with constant coefficients can be written in the form

$$\boxed{y'' + ay' + by = f(t)}$$

where a, b are constants, f(t) is a function in t.

(a) Homogeneous case: $f(t) = 0$

$$\boxed{y'' + ay' + by = 0}$$

Ex: $y'' - 3y' + 2y = 0$

Char eqn: $r^2 - 3r + 2 = 0$

Char. roots $\longrightarrow$ $r = 1$, $r = 2$

$$r = \frac{3 \pm \sqrt{9 - 4 \cdot 2}}{2} = \frac{3 \pm 1}{2} = 2, 1$$

$\Downarrow$

$$y = C_1 \cdot e^t + C_2 e^{2t}$$

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Fact: A second order diff. eqn. will have a general solution depending on two unknown constants C_1, C_2.

$$y'' = 6t \Rightarrow y' = 3t^2 + C \Rightarrow y = \int 3t^2 + C\,dt$$
$$= t^3 + Ct + D$$

Linear homogeneous case:

When is $y = e^{rt}$ a solution?

$$y = e^{rt} \quad y' = e^{rt} \cdot r$$
$$y'' = e^{rt} \cdot r^2$$

$$y'' + ay' + by = 0$$
$$r^2 e^{rt} + a \cdot r e^{rt} + b \cdot e^{rt} = 0$$
$$(r^2 + ar + b)e^{rt} = 0$$
$$\boxed{r^2 + ar + b = 0}$$

char. eqn.

The char. roots = solutions of char. eqn. $r^2 + ar + b = 0$ are exactly the values of r s.t. $y = e^{rt}$ is a solution of $y'' + ay' + by = 0$.

r is a solution of $r^2 + ar + b = 0$ $\Leftrightarrow$ $y = e^{rt}$ is a solution of $y'' + ay' + by = 0$

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General case: $y'' + ay' + by = 0$

Char. eqn: $r^2 + ar + b = 0$

$$r = \frac{-a \pm \sqrt{a^2-4b}}{2}$$

i) $a^2 - 4b > 0$: $r_1 = \frac{-a+\sqrt{a^2-4b}}{2}$ $r_2 = \frac{-a-\sqrt{a^2-4b}}{2}$

two distinct roots $r_1 \neq r_2$

General solution:

$$y = c_1 \cdot e^{r_1 t} + c_2 \cdot e^{r_2 t}$$

ii) $a^2 - 4b = 0$: $r = -\frac{a}{2}$ double root (mult. 2)

General solution:

$$y = c_1 e^{rt} + c_2 t e^{rt}$$
$$= (c_1 + c_2 t) e^{rt}$$

iii) $a^2 - 4b < 0$: No real solutions

$$r = \frac{-a \pm \sqrt{a^2-4b}}{2} = \frac{-a \pm \sqrt{4b-a^2} \cdot \sqrt{-1}}{2}$$

$$= -\frac{a}{2} \pm \boxed{\frac{\sqrt{4b-a^2}}{2}} \cdot \sqrt{-1}$$

General solution: $\alpha = -a/2$, $\beta = \sqrt{4b-a^2}/2$

$$y = e^{\alpha t} \cdot (c_1 \cdot \cos(\beta t) + c_2 \cdot \sin(\beta t))$$

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Ex: $y'' - 4y' + 4y = 0$

$r^2 - 4r + 4 = 0$

$r = \frac{4 \pm \sqrt{16 - 4 \cdot 4}}{2} = \frac{4}{2} = \underline{2}$ double root

$y = C_1 \cdot e^{2t} + C_2 \cdot t \cdot e^{2t} = \underline{\underline{(C_1 + C_2 t) e^{2t}}}$

$\left(C_1 e^{2t} + C_2 e^{2t} = (C_1 + C_2) \cdot e^{2t} = K \cdot e^{2t} \right)$

Ex: $y'' + y = 0$ $\quad a = 0 \quad b = 1$

$r^2 + 1 = 0$

$r^2 = -1$

$r = \underline{\pm \sqrt{-1}}$

$r^2 + 1 = 0$

$r = \frac{0 \pm \sqrt{0 - 4}}{2} = \frac{0 \pm \sqrt{-4}}{2} = \frac{0 \pm \sqrt{4} \cdot \sqrt{-1}}{2}$

$= \underline{0 \pm 1 \cdot \sqrt{-1}}$

$\alpha = 0 \quad \beta = 1$

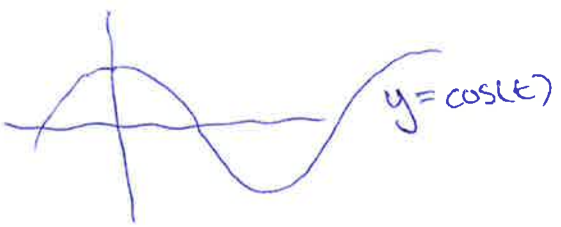
$y = e^{0 \cdot t} \cdot (C_1 \cdot \cos(1 \cdot t) + C_2 \cdot \sin(1 \cdot t))$

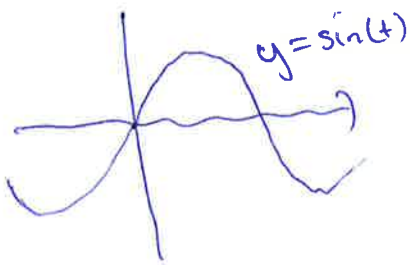
$= \underline{\underline{C_1 \cos(t) + C_2 \sin(t)}}$

Formula:

$\alpha = -a/2 = 0$

$\beta = \frac{\sqrt{4b - a^2}}{2} = \frac{2}{2} = 1$





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b) <u>Inhomogeneous case:</u> $y'' + ay' + by = f(t)$

<u>Super-position principle:</u>

The general solution of the inhomogeneous diff.eqn. $y''+ay'+by=f(t)$ is

$$y = y_h + y_p$$

where

(1) y_h is the general solution of the homogeneous equation $y''+ay'+by=0$

(2) y_p is a <u>particular</u> solution of inhomogeneous diff.eqn. $y''+ay'+by=f(t)$.

<u>Ex:</u> $y'' - 4y' + 7y = 3$

General solution: $y = y_h + y_p =$

$$y_h = \underline{e^{2t} \cdot (C_1 \cos\sqrt{3}t + C_2 \sin\sqrt{3}t)}$$

<u>y_h:</u> $y''-4y'+7y=0$

$$r^2 - 4r + 7 = 0$$

$$r = \frac{4 \pm \sqrt{16-4\cdot 7}}{2}$$

$$= 2 \pm \frac{\sqrt{-12}}{2} = 2 \pm \sqrt{-1}\cdot\frac{\sqrt{12}}{2}$$

$$= 2 \pm \sqrt{-1}\cdot\frac{\cancel{\sqrt{4}}\cdot\sqrt{3}}{\cancel{2}} = 2 \pm \sqrt{-1}\cdot\sqrt{3}$$

$$\underline{\alpha = 2}, \quad \beta = \underline{\sqrt{3}}$$

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y_p: $$y'' - 4y' + 7y = \underbrace{3}_{f(t)}$$ need to find one solution

Guess: $y = A$ (constant)
$y' = 0$
$y'' = 0$

$$0 - 4 \cdot 0 + 7 \cdot A = 3$$
$$7A = 3$$
$$A = 3/7$$

$$y_p = 3/7$$

Conclusion: $$y = y_h + y_p = e^{2t}\left(C_1 \cos(\sqrt{3}t) + C_2 \sin(\sqrt{3}t)\right) + 3/7$$

Ex: $$y'' - 3y' + 2y = e^{-t}$$

$$y = y_h + y_p = C_1 e^{t} + C_2 e^{2t} + \dots$$

y_p: Guess:
- should contain constants that you can vary
- should have the same form as $f(t) \longrightarrow$ Compute f', f''

$$f(t) = e^{-t} \longrightarrow Ae^{-t}, \; e^{At}, \; \dots$$
$$f'(t) = e^{-t} \cdot (-1) = -e^{-t}$$
$$f''(t) = e^{-t}$$

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$$y'' - 3y' + 2y = e^{-t}$$

$$\begin{cases} y = A\cdot e^{-t}, \quad A \text{ const} \\ y' = -Ae^{-t} \\ y'' = Ae^{-t} \end{cases}$$

$$Ae^{-t} - 3(-Ae^{-t}) + 2Ae^{-t} = e^{-t}$$

$$(A + 3A + 2A)e^{-t} = e^{-t}$$

$$A + 3A + 2A = 1$$

$$6A = 1$$

$$\underline{A = 1/6} \longrightarrow y_p = \frac{1}{6}e^{-t}$$

If the initial guess doesn't work, try to multiply it with $\underline{t}$.

General solution: $y = \underline{\underline{C_1 e^{t} + C_2 e^{2t} + \frac{1}{6}e^{-t}}}$

Try to solve: $y'' - 3y' + 2y = te^{-t}$

Hint: For y_p, try $y = (At+B)e^{-t}$

A, B constants

$$\begin{cases} f(t) = te^{-t} \\ f'(t) = 1\cdot e^{-t} + t\cdot e^{-t}\cdot(-1) \\ \quad = (1-t)e^{-t} \\ \vdots \end{cases}$$