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LECTURE 10

GRA 6035

MATHEMATICS

EIVIND ERIKSEN

OCT 27, 2016

Plan:

1. Differential equations
2. First order differential equations (- seperable - linear - exact) ← next lecture (exact)

Reading:

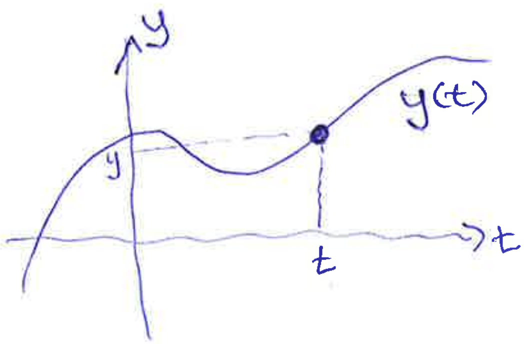
[MEJ] 24.1-24.2, (24.4-24.6)

Note: Integration

1 Differential equations

Ex:

$$y'(t) = 2t-4 \qquad y' = 2t-4$$
$$y'(t) = 6y(t) \qquad y' = 6y$$
$$y'(t)\cdot y(t) = y(t)^2 - y(t) \qquad y'y = y^2 - y$$



$y' = 6y$

$t=3, y=1$: $y' = 6\cdot 1 = 6$

A differential equation is an equation in $y(t)=y$ that contains $y'(t)=y'$ (and possibly higher order derivatives $y''(t)=y''$, $y'''(t)=y'''$, ...).

Order = highest order derivative that appears.

A solution is a function $y(t)$ that fits in the diff. eqn.

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Ex: $y'(t) = 2t-4$ $\qquad y' = 2t-4$

$$\int y'(t)\,dt = \int 2t-4\,dt$$

$$y(t) = \int 2t-4\,dt$$

$$y(t) = \underline{\underline{t^2-4t+C}}$$ general solution of the differential eqn.

$y' = 2t-4$, $y(0)=1$ initial condition

$t=0$, $y=1$

$y = t^2-4t+C$

$1 = 0-0+C \qquad \underline{C=1}$

[ME] Integration A4

Sydsæter: Essential Math Analysis

Eriksen: Matematikk

Review: Integration

$$\int f(t)\,dt = F(t)+C,$$ where $F(t)$ is an anti-derivative of $f(t)$, i.e. $F'(t) = f(t)$

Integration rules:

$$\int t^n\,dt = \frac{1}{n+1}t^{n+1} + C$$ (for all real numbers $n \neq -1$)

$$\int \frac{1}{t}\,dt = \ln|t| + C$$

$$\int e^t\,dt = e^t + C$$

$$\int (u(t) \pm v(t))\,dt = \int u(t)\,dt \pm \int v(t)\,dt$$

$$\int c\cdot u(t)\,dt = c\cdot \int u(t)\,dt$$ when c is a constant

Ex: $$\int t^2-3t+4\,dt = \frac{1}{3}t^3 - 3\cdot\frac{1}{2}t^2 + 4t + C$$

$$= \underline{\frac{1}{3}t^3 - \frac{3}{2}t^2 + 4t + C}$$

$$\int \frac{t^2-1}{t}\,dt = \int t - \frac{1}{t}\,dt = \underline{\frac{1}{2}t^2 - \ln|t| + C}$$

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Integration techniques

1. Integration by parts:

$$\int u'v\,dt = uv - \int uv'\,dt$$

Ex: $\int t\cdot \ln t\,dt = \frac{1}{2}t^2\cdot \ln t - \int \frac{1}{2}t^2\cdot\frac{1}{t}\,dt$

$$\boxed{\begin{matrix} u'=t & v=\ln t \\ u=\frac{1}{2}t^2 & v'=1/t \end{matrix}}$$

$$= \frac{1}{2}t^2\cdot \ln t - \int \frac{1}{2}t\,dt = \frac{1}{2}t^2\ln t - \frac{1}{2}\cdot\frac{1}{2}t^2 + C$$

$$= \frac{1}{2}t^2\ln t - \frac{1}{4}t^2 + C$$

2. Substitution:

Ex: $\int \sqrt{t^2+1}\,dt = \int \sqrt{u}\cdot\frac{du}{2t} = \int \sqrt{u}\cdot\frac{1}{2t}\,du$ ↑ difficult

$$\boxed{\begin{matrix} u=t^2+1 \\ du=u'\,dt \\ du=2t\,dt \end{matrix}}$$

$\rightarrow t^2 = u-1 \quad t=\pm\sqrt{u-1}$

$\rightarrow dt = \frac{du}{2t}$

$$\int \sqrt{t^2+1}\cdot 2t\,dt = \int \sqrt{u}\cdot \cancel{2t}\cdot\frac{du}{\cancel{2t}}$$

$$\boxed{\begin{matrix} u=t^2+1 \\ du=2t\,dt \end{matrix}}$$

$$= \int\sqrt{u}\,du = \int u^{1/2}\,du = \frac{2}{3}\cdot u^{3/2} + C$$

$$= \frac{2}{3}(t^2+1)^{3/2} + C$$

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(3) Integration of rational functions

Ex: $\int \frac{t^2}{t^2-3t+2}\,dt = \int 1 + \frac{3t-2}{t^2-3t+2}\,dt = t + \int \frac{3t-2}{t^2-3t+2}\,dt$

Polynomial division:

$t^2 : (t^2-3t+2) = 1$
$-(t^2-3t+2)$
$3t-2$

$$\frac{t^2}{t^2-3t+2} = 1 + \frac{3t-2}{t^2-3t+2}$$

Simplify $\frac{3t-2}{t^2-3t+2}$:

$t^2-3t+2 = (t-1)(t-2)$
(since $t^2-3t+2=0$ gives $t=1,2$)

$(t-1)(t-2)\cdot \Big| \quad \frac{3t-2}{(t-1)(t-2)} = \frac{\overset{=-1}{A}}{t-1} + \frac{\overset{=4}{B}}{t-2}$ ← Find constants A, B

$(3t-2) = A(t-2) + B(t-1)$

Alt (A)

$3t-2 = A(t-2) + B(t-1)$
$= At - 2A + Bt - B$
$3t-2 = (A+B)t + (-2A-B)$

$A+B=3$
$-2A-B=-2$
$-A = 1$

$A=-1$
$B=4$

Alt (B)

$3t-2 = A(t-2)+B(t-1)$

$t=2$: $4 = B\cdot 1$ $\quad B=4$
$t=1$: $1 = A\cdot(-1)$ $\quad A=-1$

$$\int \frac{t^2}{t^2-3t+2}\,dt = t + \int \frac{3t-2}{t^2-3t+2}\,dt$$
$$= t + \int \frac{-1}{t-1} + \frac{4}{t-2}\,dt$$
$$= t - \ln|t-1| + 4\cdot \ln|t-2| + C$$

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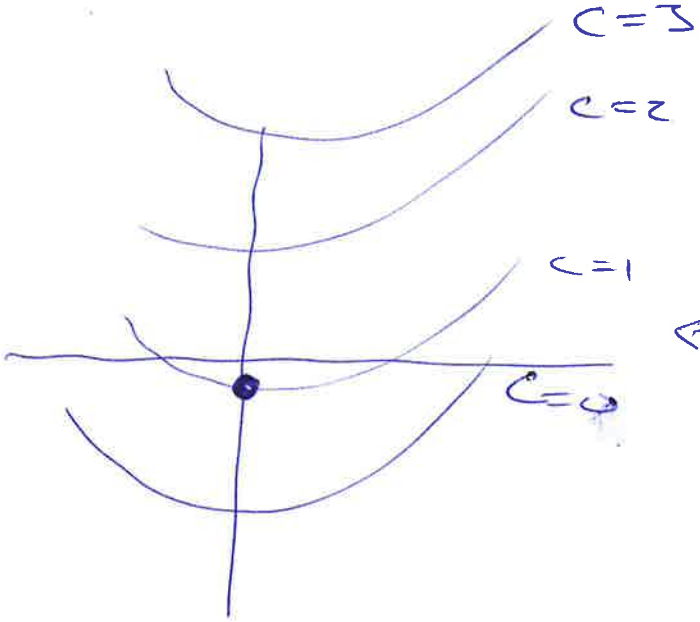
$y' = 2t - 4 \longrightarrow y(t) = \underline{t^2 - 4t + C}$

general solution

$y' = 2t - 4,\ y(0) = 1 \longrightarrow C = 1,\ y = t^2 - 4t + 1$

particular solution



a differential equation will have many different solutions $\Longleftrightarrow$ different values of C

In general: First order differential equations

$$y' = F(y, t)$$

have general solution with one undetermined coeff. C.

If there is an initial condition, that can be used to determine C.

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2) First order differential equations

(a) Seperable: $\boxed{y' = f(y) \cdot g(t)}$ ← seperable: can be written in this form

Ex: ① $yy' = 2t$ seperable

$$y' = \underbrace{\frac{1}{y}}_{f(y)} \cdot \underbrace{2t}_{g(t)} = \frac{2t}{y}$$

② $y' = y + t$ not seperable

Method: $y' = \frac{1}{y} \cdot 2t \quad | \cdot y$

$$yy' = 2t$$

$dy = y'dt$ ⟶ $$\int y \underbrace{y' dt}_{dy} = \int 2t\,dt$$

$$\int y\,dy = \int 2t\,dt$$

$$\tfrac{1}{2}y^2 = t^2 + (C_2 - C_1) \longleftarrow \tfrac{1}{2}y^2 + C_1 = t^2 + C_2$$

$C = C_2 - C_1$

$K = 2C$

$$y^2 = 2t^2 + 2C$$

$$y = \pm\sqrt{2t^2 + 2C}$$

general solution

$$y' = f(y) \cdot g(t)$$

$$\frac{1}{f(y)} y' = g(t)$$

$$\int \frac{1}{f(y)} \underbrace{y' dt}_{dy} = \int g(t)\,dt$$

$$\boxed{\int \frac{1}{f(y)}\,dy = \int g(t)\,dt}$$

implicit solution → $y^2 = 2t^2 + 2C$

explicit solution

$y = \ldots$

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$yy' = 2t,\ y(0) = 1$:

$yy' = 2t$

$\Downarrow$

$y = \pm\sqrt{2t^2 + 2c}$

$y(0) = 1$: $t = 0$, $y = 1$

$1 = +\sqrt{0 + 2c}$

$1 = \sqrt{2c}$

$1 = 2c$

$c = 1/2$

$\Downarrow$

Particular solution:

$y = \sqrt{2t^2 + 1}$

Ex: $y' = 2y,\ y(0) = 10$

$y' = y \cdot 2$

$\quad\ \ f(y) \quad g(t)$

$\frac{1}{y} y' = 2$

$\int \frac{1}{y} y' \, dt = \int 2 \, dt$

$\int \frac{1}{y} \, dy = \int 2 \, dt$

$\ln|y| = 2t + C$ implicit solution

$e^{\ln|y|} = e^{2t+C}$

$|y| = e^{2t+C} = e^{2t} \cdot e^{C}$

$y = \pm e^{2t} e^{C}$

general solution $\longrightarrow$ $y = Ke^{2t}$ $\quad K = \pm e^{C}$

$y = Ke^{2t}$

$10 = Ke^{2 \cdot 0} = K$

$K = 10$

$y = 10e^{2t}$

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6 Linear first order differential equations

$$\boxed{y' + a(t)\cdot y = b(t)} \iff y' = b(t) - a(t)\cdot y$$

linear: can be written in this form

Ex: $\begin{cases} y' = y + t \\ y' - y = t \end{cases}$ not separable

linear $\left(a(t) = -1 \;,\; b(t) = t \right)$

Method: Integrating factor

Integrating factor:

$$u = e^{\int a(t)\,dt}$$

$$a(t) = -1$$

$$\int a(t)\,dt = -t + C$$

$$\boxed{u = e^{-t}}$$

Ex: $y' - y = t \quad | \cdot e^{-t}$

$$\underbrace{y'e^{-t} - ye^{-t}}_{(y\cdot e^{-t})'} = te^{-t}$$

$$(y\cdot e^{-t})' = y'\cdot e^{-t} + y\cdot e^{-t}\cdot(-1)$$

$$(y\cdot e^{-t})' = te^{-t}$$

$$\int (y\cdot e^{-t})'\,dt = \int te^{-t}\,dt$$

$$\int u'v\,dt = uv - \int uv'\,dt$$

Integration by parts

$$y\cdot e^{-t} = \int te^{-t}\,dt = -e^{-t}\cdot t - \int -e^{-t}\cdot 1\,dt$$

$$\begin{array}{ll} v = t & u' = e^{-t} \\ v' = 1 & u = -e^{-t} \end{array}$$

$$ye^{-t} = -te^{-t} + \int e^{-t}\,dt = -te^{-t} - e^{-t} + C$$

$$y = e^{t}\cdot\left(-te^{-t} - e^{-t} + C\right)$$

$$= -t - 1 + Ce^{t}$$

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In general:

$$y' + a(t)\,y = b(t)$$

$$(y\cdot u)' = b(t)\cdot u(t)$$

$$y\cdot u(t) = \int b(t)\cdot u(t)\,dt$$

$$y = \frac{1}{u(t)}\cdot \int b(t)\,u(t)\,dt$$

$$u = e^{\int a(t)\,dt}$$

integrating factor

Special case:

$a(t) = a$ constant

$$\int a(t)\,dt = \int a\,dt = at + C$$

$$u = e^{at}$$

$$\boxed{y = e^{-at}\int b(t)e^{at}\,dt = \dots + Ce^{-at}}$$

Ex:

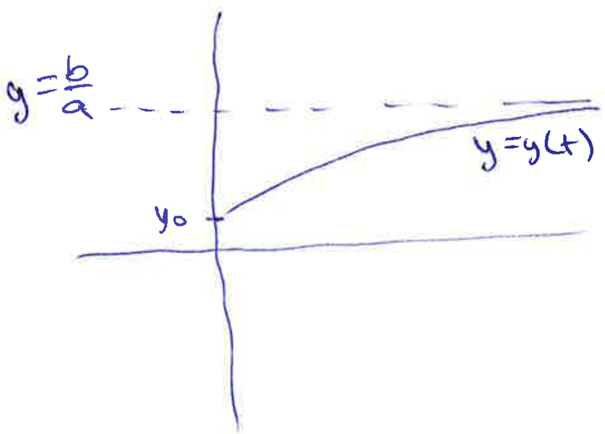
$$y' + ay = b$$

$$y = e^{-at}\cdot\int b e^{at}\,dt$$

$$= e^{-at}\left(b\cdot e^{at}\cdot \frac{1}{a} + C\right)$$

$$\left(e^{at}\cdot\frac{1}{a}\right)' = e^{at}\cdot \cancel{a}\cdot\frac{1}{\cancel{a}} = e^{at}$$ Substitution

$$\boxed{y = \frac{b}{a} + Ce^{-at}}$$



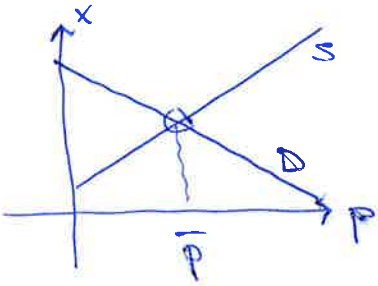
$$\lim_{t\to\infty} y(t) = \frac{b}{a} + \lim_{t\to\infty} Ce^{-at} = \frac{b}{a}$$

case $a>0$:

Ex:

$$D = a - bp \quad \text{(demand)}$$
$$S = \alpha + \beta p \quad \text{(supply)}$$

$a, b, \alpha, \beta > 0$ constants

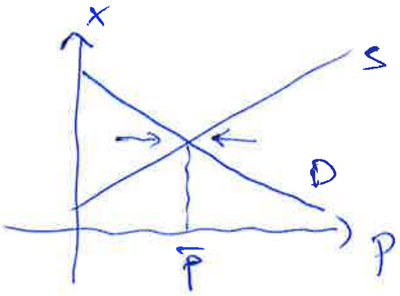


$$p' = \lambda \cdot (D - S), \quad \lambda > 0 \text{ constant}$$

(differential equation modelling how quickly and in which direction p will change

$D > S$: $p' > 0$ $\rightarrow$ (p increases)

$D < S$: $p' < 0$ $\leftarrow$ (p decreases)



$$a - b\bar{p} = \alpha + \beta \bar{p}$$
$$a - \alpha = (b + \beta)\bar{p}$$
$$\boxed{\bar{p} = \frac{a - \alpha}{b + \beta}}$$

equilibrium price

Solution:

$$p' = \lambda \cdot (D - S) = \lambda \cdot ((a - bp) - (\alpha + \beta p))$$
$$= \lambda \cdot ((a - \alpha) - (b + \beta)p)$$

$$p' + \underbrace{\lambda \cdot (b + \beta)}_{\text{pos. const.}} p = \underbrace{\lambda (a - \alpha)}_{\text{pos. const.}}$$

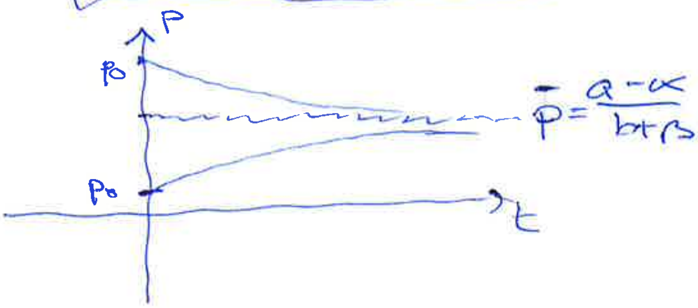
$$\boxed{\begin{aligned} &y' + ay = b \\ &\Downarrow \\ &y = \frac{b}{a} + C \cdot e^{-at} \end{aligned}}$$

$$p = \frac{\lambda (a - \alpha)}{\lambda (b + \beta)} + C \cdot e^{-\lambda (b + \beta) t}$$

$$\boxed{p = \frac{a - \alpha}{b + \beta} + C \cdot e^{-\lambda (b + \beta) t}}$$

$\bar{p} = \dfrac{a - \alpha}{b + \beta}$ eq. price

with globally asympt. stable price



Solution gives precisely how the price will develope over time and approach $\bar{p}$.

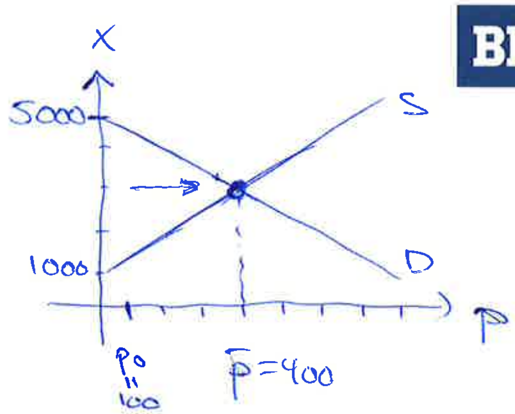
Numerical example:

$D = 5000 - 4p$

$S = 1000 + 6p$

$p' = \frac{1}{2}(D - S)$

$\bar{p} = \frac{4000}{10} = \underline{400}$



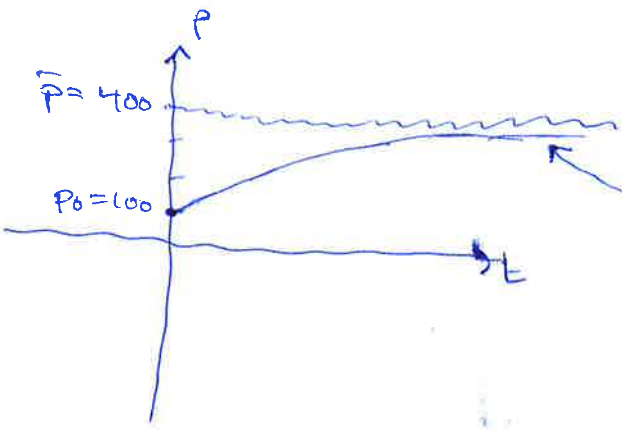
$$p(t) = \underline{400 + Ce^{-5t}}$$

$\underline{p(0) = 100}$

$400 + C \cdot e^{-5 \cdot 0} = 100$

$400 + C = 100$

$C = -300$



$p(t) = 400 - 300e^{-5t}$

Postponed for next lecture:

c) Exact differential equations