Review Lecture 3

GRA 6035 MATHEMATICS BI

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Plan: Review: Optimization in several variables

Key concepts:

Unconstrained Optimization:

$f(x_1, \ldots, x_n)$

- * find partial derivatives, compute Hessian matrix
- * find stationary pts of f and classify them as local max/min or saddle pts.
- * convex/concave property
- * global max/min.
- * envelope thms.

Stationary pts:

$$f'_x = f'_y = f'_z = \dots = 0$$

Remember:

$$f(\underline{x}) = \ln(\underline{u}(x)) \Rightarrow f'_x = \frac{1}{u} \cdot u'_x$$

$$f'_y = \frac{1}{u} \cdot u'_y$$

$$\vdots$$

$$g(x) = e^{u(\underline{x})} \Rightarrow g'_x = e^{u(\underline{x})} \cdot u'_x$$

$$g'_y = e^{u(\underline{x})} \cdot u'_y$$

$$\vdots$$

Classification: $\underline{x}^*$ stationary pt.

$H(f)(\underline{x}^*) \xrightarrow{\text{definiteness}}$

- positive defn. $\Rightarrow \underline{x}^*$ local min
- negative defn. $\Rightarrow \underline{x}^*$ local max
- indefinite $\Rightarrow \underline{x}^*$ saddle pt.

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Global max/min:

f is <u>concave</u> $\Leftrightarrow$ $H(f)(\underline{x})$ is neg. semidefn. for all $\underline{x}$

f is <u>convex</u> $\Leftrightarrow$ $H(f)(\underline{x})$ is pos. semidefn. for all $\underline{x}$

If f is concave (convex), then any stationary pt. is a global <u>max</u> (min)

<u>EVT:</u> If f is cont. and defined on a closed and bounded set, then f has a global max and a global min.

<u>Ex:</u> $f(x,y) = \sqrt{x^2+y^2-xy}$ $\quad D_f = \{(x,y) : x^2+y^2-xy \geq 0\}$

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Problem:

$f(x,y,z) = e^{xy + yz - xz} = e^u$

i) Find stationary pts

ii) Classify them as local max/min or saddle pt.

iii) What about global max/min?

$e^u > 0$

$f'_x = e^u \cdot \underbrace{(y-z)}_{u'_x} = 0 \longrightarrow y - z = 0 \quad y = z$

$f'_y = e^u \cdot (x+z) = 0 \qquad x + z = 0 \quad 2x = 0$

$f'_z = e^u \cdot (y-\cancel{x}) = 0 \qquad y - \cancel{x} = 0 \quad x = 0, \; x = y$

$\Downarrow$

$(x,y,z) = \underline{(0,0,0)}$

Stationary pt.

H(f):

$f''_{xx} = \underline{e^u (y-z)^2}$

$f''_{xy} = e^u \cdot (x+z) \cdot (y-z) + 1 \cdot e^u$
$= e^u \cdot ((x+z)(y-z) + 1)$

$f''_{yy} = \underline{e^u \cdot (x+z)^2}$

$f''_{xz} = e^u (y-x) \cdot (y-z) + (-1) \cdot e^u$
$= e^u \cdot ((y-x)(y-z) - 1)$

$f''_{zz} = e^u \cdot (y-x)^2$

$f''_{yz} = e^u (y-x)(x+z) + 1 \cdot e^u$
$= e^u \cdot ((y-x)(x+z) + 1)$

$$H(f) = \begin{pmatrix} (y-z)^2 & 1+(x+z)(y-z) & -1+(y-x)(y-z) \\ 1+(x+z)(y-z) & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \cdot e^u$$

$$H(f)(0,0,0) = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \cdot e^0$$

$$H(f)(0,0,0) = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

$D_1 = 0$

$D_2 = -1$

indefinite

Remember: $D_2 < 0 \rightarrow$ indefinite

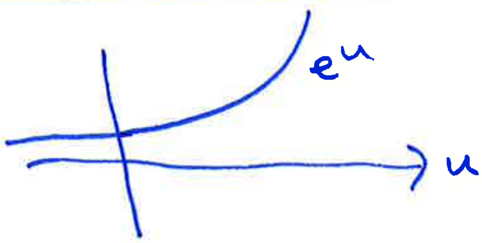
$(D_4 < 0)$

D_1, D_3 opposite signs $\rightarrow$ indefinite

$\Downarrow$

$(0,0,0)$ saddle pt.

Global max/min?



$f = e^u \qquad u = xy + yz - xz$

f increases when u increases

no max
no min
for u

$$A = \begin{pmatrix} 0 & 1/2 & -1/2 \\ 1/2 & 0 & 1/2 \\ -1/2 & 1/2 & 0 \end{pmatrix}$$

Constrained optimization

* Lagrange problems ← equality constr.
* Kuhn-Tucker problems ← inequality constr.

Ex: max $x^2+5y^2+3z^2$ when $\begin{cases} x^2+y^2+z^2 \le 1 \\ 2x-y+4z \le 5 \end{cases}$

- how to solve the problem
  - candidate pts: { Lagrange conditions / KT - conditions
  - determine if candidate pts are max/min.
    - * SOC (second order condition)
    - * EVT (is the set of adm. pts. closed and bounded) + exhaustion
- envelope thms.

Problem: $\max \underset{\substack{\shortparallel \\ f(x,y,z)}}{x^2+5y^2+3z^2}$ when $\begin{cases} \overset{g_1}{\shortparallel}\; x^2+y^2+z^2 \le 1 \;\overset{a_1}{\shortparallel} \\ \underset{g_2}{\shortparallel}\; 2x-y+4z \le 5 \;\underset{a_2}{\shortparallel} \end{cases}$

ok.

ok.

√ KT: std form $= \max + g \le a$

Candidate pts: $\mathcal{L} = x^2+5y^2+3z^2 - \lambda_1(x^2+y^2+z^2) - \lambda_2 \cdot (2x-y+4z)$

KT-conditions

FOC: $\mathcal{L}'_x = 2x - \lambda_1 \cdot 2x - \lambda_2 \cdot 2 = 0$

$\mathcal{L}'_y = 10y - \lambda_1 \cdot 2y + \lambda_2 = 0$

$\mathcal{L}'_z = 6z - \lambda_1 \cdot 2z - \lambda_2 \cdot 4 = 0$

C: $x^2+y^2+z^2 \le 1$

$2x-y+4z \le 5$

CSC: $\lambda_1 \ge 0$ and $\lambda_1 \cdot (x^2+y^2+z^2-1) = 0$

$\lambda_2 \ge 0$ " $\lambda_2 \cdot (2x-y+4z-5) = 0$

Alt:

$\lambda_1 = 0$ if $x^2+y^2+z^2 < 1$ (non-bind.)

$\lambda_2 = 0$ if $2x-y+4z < 5$ (non-bind.)

1) $2x\cdot(1-\cancel{1}\lambda_1) = 2\lambda_2$

$x = \frac{\cancel{2}\lambda_2}{\cancel{2}(1-\lambda_1)}$ or $\lambda_1 = 1$

$x = \frac{\lambda_2}{1-\lambda_1}$ or $\lambda_1=1,\ \lambda_2=0$ A

2) $2y(5-\lambda_1) = -\lambda_2$

$y = \frac{-\lambda_2}{2(5-\lambda_1)}$ or $\lambda_1=5,\lambda_2=0$ B

3) $\cancel{4}z(3-\lambda_1) = \cancel{4}^{2}\lambda_2$

$z = \frac{2\lambda_2}{3-\lambda_1}$ or $\lambda_1=3,\lambda_2=0$ C

Cases:

A) $\lambda_1=1,\lambda_2=0$: $y=0, z=0$

$\lambda_1\cdot(x^2+y^2+z^2-1)=0 \Rightarrow x^2+y^2+z^2=1 \Rightarrow x=\pm 1$

$(\pm1,0,0;1,0)$ $f=1$

B) $\lambda_1=5,\lambda_2=0$: $x=0, z=0, y=\pm1$

$(0,\pm1,0;5,0)$ $f=5$

c) $\underline{\lambda_1 = 3, \lambda_2 = 0}$: $x = 0, y = 0, z = \pm 1$

$\underline{(0, 0, \pm 1; 3, 0)}$ $f = 3$

I will try to check $(x, y, z; \lambda_1, \lambda_2)$
$= (0, \pm 1, 0; 5, 0)$

using SOC even
if there are more cases
to check for candidates.

$\underline{\text{SOC}}$: $(0, \pm 1, 0; 5, 0)$

$$\mathcal{L}(x, y, z; 5, 0) = x^2 + 5y^2 + 3z^2 - 5(x^2 + y^2 + z^2) - 0 \cdot (2x - y + 4z)$$

$$= \underline{-4x^2 - 2z^2} \leftarrow \text{is it concave?}$$

$$H = \begin{pmatrix} -8 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

$D_1 = -8$ $\Delta_1 = -8, 0, -4 \leq 0$

$D_2 = 0$ $\Delta_2 = 0, 0, 32 \geq 0$

$D_3 = 0$ $\Delta_3 = 0 \leq 0$

ok., it is $\underline{\text{concave}}$

$\underline{\text{Conclusion:}}$

$(x, y, z) = (0, \pm 1, 0)$ $\underline{\text{are}}$ max by SOC

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Alternatives: If SOC fails

(1) Adm pts: All pts that satisfy all constraints

$$x^2+y^2+z^2 \leq 1$$

$$2x-y+4z \leq 5$$

Is the set of adm. pts bounded?

Answer: $-1 \leq x \leq 1$ $\Rightarrow$ the set is bounded.

$-1 \leq y \leq 1$

$-1 \leq z \leq 1$

$\Downarrow$ EVT

there is a max.

To find the max this way, we would have to compute all candidate pts (all cases) + pts where NDCQ fails.