REVIEW LECTURE 2

GRA 6035 MATHEMATICS

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Plan: Review: Differential equations

18p

Key concepts:

* integration
* linear diff. eqn.
* seperable "
* exact "

a) Integration:

$$\int t^n \, dt = \frac{1}{n+1} t^{n+1} + C$$

$$\int e^t \, dt = e^t + C \qquad \int 1/t \, dt = \ln|t| + C$$

Substitution: $\int f(u) \cdot u' \, dt = \int f(u) \, du$

Ex: $\int t e^{-t^2} \, dt$

$u = -t^2$

$du = -2t \cdot dt$ $\quad u'$

$\boxed{du = u' \cdot dt}$

$$= \int \cancel{t} e^u \cdot \frac{du}{-2\cancel{t}}$$

$$= \int e^u \cdot \left(-\tfrac{1}{2}\right) du = -\tfrac{1}{2} \cdot e^u + C = \underline{-\tfrac{1}{2} e^{-t^2} + C}$$

Integration by parts:

$$\int u'\cdot v\,dt = uv - \int uv'\,dt$$

Use this to integrate products.

$$\cancel{\left(\int uv'\,dt = uv - \int u'v\,dt\right)}$$

Ex: $\int \overset{v}{t}\,\overset{u'}{e^{-t}}\,dt =$ int. by parts!

$$\underset{v'}{1}\cdot\underset{u}{(-e^{-t})} \qquad \cancel{\tfrac{1}{2}t^2\cdot(-e^{-t})}$$

$$= \overset{u\cdot v}{\underline{-e^{-t}\cdot t}} - \int \overset{uv'}{-e^{-t}\cdot 1}\,dt$$

$$= -te^{-t} + \int e^{-t}\,dt = \underline{-te^{-t} - e^{-t} + C}$$

b) Linear differential equations.

$$y'' + ay' + by = f(t) \Rightarrow y = y_h + y_p$$

y_h: $r^2 + ar + b = 0$ $\Rightarrow$ $y_h = \underline{C_1\cdot e^{r_1 t} + C_2 e^{r_2 t}}$

$\underline{r = r_1}$, $\underline{r = r_2}$ $\quad r_1 \neq r_2$

$\underline{y_p}$: Make a guess from the form of $f(t)$: $y(t)$ and put it into the diff. eqn.

$\underline{\text{Problem:}}$ $y'' - 3y' - 10y = e^{2t}$.

Solution: $y'' - 3y' - 10y = e^{2t}$ linear

$y = y_h + y_p$

y_h: $r^2 - 3r - 10 = 0$

$r = \frac{3 \pm \sqrt{9 - 4(-10)}}{2} = \frac{3 \pm 7}{2} = 5, -2$

$y_h = C_1 \cdot e^{5t} + C_2 e^{-2t}$

y_p: e^{2t} $\quad (e^{2t})' = e^{2t} \cdot 2 \quad (e^{2t})'' = 4e^{2t}$

Guess: $y = A \cdot e^{2t}$

$y' = 2Ae^{2t}$

$y'' = 4Ae^{2t}$

$y'' - 3y' - 10y = e^{2t}$

$(4Ae^{2t}) - 3 \cdot (2Ae^{2t}) - 10(Ae^{2t}) = e^{2t}$

$(4A - 6A - 10A)e^{2t} = e^{2t}$

$-12A = 1 \quad A = -1/12 \quad y_p = -\frac{1}{12}e^{2t}$

$y = c_1 e^{5t} + c_2 e^{-2t} - \frac{1}{12} e^{2t}$

b) <u>Seperable</u>

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$$y' = 3\sqrt{t} \cdot e^{-2y} \quad \Big| \cdot e^{2y}$$

$$\underline{e^{2y}} \cdot y' = \underline{3\sqrt{t}} \qquad \text{separable}$$

$$\;\; y \qquad\qquad\quad t$$

$$\boxed{\int e^{2y}\,dy = \int 3\sqrt{t}\,dt} \Leftarrow \int e^{2y} y'\,dt = \underline{\int 3\sqrt{t}\,dt}$$

$$\int e^{2y}\,dy = \int 3t^{1/2}\,dt \qquad \int t^n\,dt = \frac{1}{n+1}t^{n+1} + C$$

$$\frac{1}{2}\cdot e^{2y} = 3\cdot\frac{1}{3/2}\cdot t^{3/2} + C$$

$$2\cdot\Big|\quad \frac{1}{2}e^{2y} = \frac{3}{3/2}t^{3/2} + C \qquad \underline{\text{implicit solution}}$$

$$e^{2y} = \frac{6\cdot 2}{3/\cancel{2}\cdot\cancel{2}}t^{3/2} + 2C$$

$$e^{2y} = 4t^{3/2} + 2C$$

$$2y = \ln(4t^{3/2} + 2C)$$

$$y = \frac{1}{2}\ln(4t^{3/2} + 2C)$$

Problem ① $yy' = t^2 e^y \rightarrow y' = \frac{t^2 e^y}{y}$

② $y' = t\cdot(y-1)^2$, $y(0)=3$

$t=0, y=3$

Solution

① $yy' = t^2 e^y \quad |: e^y$

$\frac{y}{e^y}\cdot y' = t^2$

$\overline{\quad y \quad} \qquad \overline{\quad t \quad}$

Seperable

$\int \frac{y}{e^y} \underbrace{y' dt} = \int t^2 dt$

$\int \frac{y}{e^y} dy = \int t^2 dt$

$\int y\cdot e^{-y} dy = \frac{1}{3}t^3 + C$

$1\cdot(-e^{-y})$ $\quad \cancel{\frac{1}{2}y^2\cdot(-e^{-y})}$

$v = y \qquad u = -e^{-y}$

$v' = 1 \qquad u' = e^{-y}$

$-ye^{-y} - \int 1\cdot(-e^{-y}) dy$

$= -ye^{-y} + \int e^{-y} dy$

$= -ye^{-y} - e^{-y} + C$

$-(y+1)e^{-y} = \frac{1}{3}t^3 + C$ implicit solution

$(y+1)e^{-y} = -\frac{1}{3}t^3 - C$ (cannot solve explicitly)

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$$y' = t \cdot (y-1)^2$$

$$\frac{1}{(y-1)^2} y' = t$$

$$\int \frac{1}{(y-1)^2}\, dy = \int t \, dt$$

Separable

$$\int \frac{1}{u^2}\, du = \frac{1}{2}t^2 + C$$

$$\boxed{\begin{aligned} u &= y-1 \\ du &= 1 \cdot dy \end{aligned}}$$

$$\int u^{-2} du = \frac{1}{-1} \cdot u^{-1} + C$$

$$= -\frac{1}{u} + C$$

$$= -\frac{1}{y-1} + C$$

$$-\frac{1}{y-1} = \frac{1}{2}t^2 + C$$

$$\frac{1}{y-1} = -\frac{1}{2}t^2 - C \quad | \cdot (y-1)$$

$$1 = (y-1)\left(-\frac{1}{2}t^2 - C\right)$$

$$\frac{1}{-\frac{1}{2}t^2 - C} = y - 1$$

$$y = \frac{1 \cdot 2}{\left(-\frac{1}{2}t^2 - C\right) \cdot 2} + 1 = \frac{2}{-t^2 - 2C} + 1$$

$$= \frac{2 - t^2 - 2C}{-t^2 - 2C}$$

d) Exact diff. eqn.

Ex: $\dfrac{t}{y^2}\, y' = \dfrac{1}{y} - 3t^2 \quad , \quad y(1) = 1/3$

$$\frac{t}{y^2}\, y' = \frac{1}{y} - 3t^2 \quad \Big| \cdot \frac{y^2}{t}$$

$$y' = \frac{y}{t} - 3t y^2$$

$$= \frac{y}{t} - 3t y^2$$

not possible to factor ans $f(y) \cdot g(t)$
$\downarrow$
not <u>seperable</u>

Exact? $\quad p + q \cdot y' = 0$

$$\left(\frac{t}{y^2}\right) \cdot y' = \frac{1}{y} - 3t^2$$

$$\underbrace{\left(3t^2 - \frac{1}{y}\right)}_{=\, h'_t} + \underbrace{\left(\frac{t}{y^2}\right)}_{=\, h'_y} \cdot y' = 0$$

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial y}\frac{dy}{dt} = 0$$
$$\Downarrow$$
$$h = C$$

$$\left.\begin{aligned} h'_t &= 3t^2 - \frac{1}{y} \\ h'_y &= \frac{t}{y^2} \end{aligned}\right\}$$

Solve for h:

possible $\rightarrow$ $h = C$

not possible $\rightarrow$ not exact

$h'_t = 3t^2 - \frac{1}{y} \Rightarrow h = \underline{t^3 - \frac{1}{y} \cdot t}$

$h'_y = \frac{t}{y^2}$

$h'_y = 0 - t\,(y^{-1})'_y$

$= -t \cdot (-1) \cdot y^{-2} = + \frac{t}{y^2}$

<u>Conclusion:</u> h satisfies both eqn.

with $h = \underline{t^3 - t/y}$

$\Downarrow$

diff. eqn. is exact, solution

(implicit solution) $h = \boxed{t^3 - t/y = C} \quad \cdot 1 \cdot y$

$yt^3 - t = C \cdot y$

$yt^3 - yC = t$

$y(t^3 - C) = t$

(explicit solution) $y = \underline{\underline{\frac{t}{t^3 - C}}}$

Problem: $(2y-e^t)y' = ye^t + 2e^{2t}$

with $y(0)=2$

$$(2y-e^t)y' = ye^t + 2e^{2t}$$

$$\underbrace{(-ye^t - 2e^{2t})}_{= h'_t} + \underbrace{(2y-e^t)}_{= h'_y}y' = 0$$

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial y}\cdot\frac{dy}{dt} = 0$$

$$\boxed{\begin{aligned} h'_t &= -ye^t - 2e^{2t} \\ h'_y &= 2y - e^t \end{aligned}}$$

$$h'_t = 0 - y\cdot e^t + \underbrace{C'(t)}_{= -2e^{2t}}$$

$$h = y^2 - e^t y + \underbrace{C(t)}_{= -e^{2t}}$$

$$C'(t) = -2e^{2t}$$

$$C(t) = \int -2e^{2t}\,dt = -e^{2t}$$

$$h = \boxed{y^2 - e^t y - e^{2t} = C}$$

diff. eqn. exact

← implicit sol.

$\underline{y(0)=2}$:

$t=0,\ y=2$

$$4 - e^0\cdot 2 - e^0 = C$$

$$4 - 2 - 1 = C$$

$$\underline{C = 1}$$

$$y^2 - e^t \cdot y - e^{2t} = 1$$

$$1 \cdot y^2 + (-e^t) \cdot y + (-e^{2t} - 1) = 0$$

$$y = \frac{e^t \pm \sqrt{e^{2t} - 4 \cdot 1 \cdot (-e^{2t} - 1)}}{2}$$

$$= \frac{e^t \pm \sqrt{5e^{2t} + 4}}{2} = \underline{\underline{\frac{e^t + \sqrt{5e^{2t} + 4}}{2}}}$$

Exam	Final exam in GRA 6035 Mathematics
Date	December 11th, 2015 at 0900 - 1200

All subquestions have the same weight and give maximal score 6p each. Answers to the first 12 subquestions give a maximal score of 72p (100%). Question 4 (c) can be skipped, but gives 6p extra credit if answered correctly.

Question 1.

We consider the matrix A given by

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

(a) **(6p)** Compute the rank of A and find all the solutions of the linear system $A \cdot \mathbf{x} = \mathbf{0}$.
(b) **(6p)** Is A diagonalizable? Find all eigenvalues of A and their multiplicities.
(c) **(6p)** Compute the eigenvalues of $B = A^2$. Is B diagonalizable?

We consider a Markov chain given by $\mathbf{x}_{t+1} = T\mathbf{x}_t$, where the transition matrix T is given by

$$T = \begin{pmatrix} 0.55 & 0.10 & 0.15 \\ 0.10 & 0.80 & 0.05 \\ 0.35 & 0.10 & 0.80 \end{pmatrix}$$

and the inital state is $\mathbf{x}_0$.

(d) **(6p)** Find the equilibrium state $\mathbf{x} = \lim_{t\to\infty} T^t \, \mathbf{x}_0$

Question 2.

Solve the differential equations:

(a) **(6p)** $y'' - 4y' - 12y = 15e^t$
(b) **(6p)** $y' = 3\sqrt{t} \cdot e^{-2y}$
(c) **(6p)** $4yt + 4t^3 + 2t + (2y - 1 + 2t^2)y' = 0, \quad y(1) = 0$

Question 3.

We consider the function given by $f(x, y, z) = \ln(u + 1)$, where $u = u(x, y, z)$ is the quadratic form

$$u(x, y, z) = 2x^2 + 2xy + 3y^2 - 2xz + z^2$$

(a) **(6p)** Determine the definiteness of u. Are there any points (x, y, z) where f is not defined?
(b) **(6p)** Compute the partial derivatives of f, and find all stationary points of f.
(c) **(6p)** Find the minimum value of f, if it exists. Is f convex?