REVIEW LECTURE 1

GRA6035 BI

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MATHEMATICS

Plan:

Review: Matrix methods

Key concepts

- Solving linear systems (Gaussian)
- Computing determinants
- Computing ranks and lin. independence
- finding eigenvalues / eigenvectors
- conditions for diagonalizability

A : $n \times n$-matrix

i) A symmetric $\Rightarrow$ A diagonalizable

ii) A diagonalizable $\Leftrightarrow$ i) there are n eigenvalues and ii) there are n lin. independent eigenvectors

$$D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

$$P = \left(\underline{v_1} \,\middle|\, \underline{v_2} \,\middle|\, \cdots \,\middle|\, \underline{v_n} \right)$$

- definiteness of a symmetric matrix

- <u>Markov chains</u>

$$\underline{x}_{t+1} = T \cdot \underline{x}_t$$

transition matrix (T), state vector ($\underline{x}_t$)

$$\underline{x} = \lim_{t\to\infty} T^t \cdot \underline{x}_0 \leftarrow \text{long run equilibrium}$$

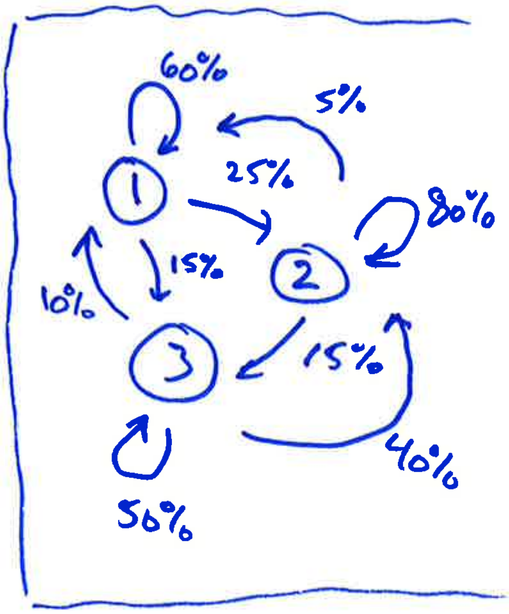
If T is a regular Markov transition matrix

(that is, i) the sum in each column of T is 1
ii) all entries in $T > 0$)

then $\underline{x}$ is the unique eigenvector with $\lambda = 1$ s.t. the sum of entries are 1.

<u>Problems:</u>

i) $T = \begin{pmatrix} 0.60 & 0.05 & 0.10 \\ 0.25 & 0.80 & 0.40 \\ 0.15 & 0.15 & 0.50 \end{pmatrix}$



Find the equilibrium state.

$\underline{\lambda = 1}$: $\begin{pmatrix} -0.40 & 0.05 & 0.10 \\ 0.25 & -0.20 & 0.40 \\ 0.15 & 0.15 & -0.50 \end{pmatrix} \cdot \underline{x} = \underline{0}$

$\frac{25}{40} = \frac{5}{8}$

$$\begin{pmatrix} -40 & 5 & 10 \\ 25 & -20 & 40 \\ 15 & 15 & -50 \end{pmatrix} \rightarrow \begin{pmatrix} -40 & 5 & 10 \\ 0 & -20+\frac{25}{8} & 40+50/8 \\ 0 & 0 & 0 \end{pmatrix} \cdot 8$$

$$\rightarrow \begin{pmatrix} -40 & 5 & 10 \\ 0 & -135 & 370 \\ 0 & 0 & 0 \end{pmatrix} \qquad -135y + 370z = 0$$

$$\begin{pmatrix} -40 & 5 & 10 \\ 0 & -135 & 370 \\ 0 & 0 & 0 \end{pmatrix} \cdot \underline{x} = \underline{0}$$

$$-135y + 370z = 0$$

$$\frac{-135y}{-135} = \frac{-370z}{-135}$$

$$y = \frac{370}{135}z = \underline{\frac{74}{27}z}$$

$$-40x + 5y + 10z = 0$$

$$-40x = -5y - 10z = -5 \cdot \frac{74}{27}z - 10z$$

$$-40x = -\frac{5 \cdot 74z}{27} - \frac{270z}{27} = \frac{-370z - 270z}{27}$$

$$-40x = \frac{-640z}{27}$$

$$x = \frac{-640z}{-40 \cdot 27} = \frac{640}{40} \cdot \frac{z}{27} = \underline{\frac{16z}{27}}$$

$$x = \frac{16z}{27} \qquad y = \frac{74}{27}z \qquad z = z \text{ (free)}$$

$\underline{x+y+z=1:}$

$$\frac{16}{27}z + \frac{74}{27}z + z = 1$$

$$z\left(\frac{16}{27} + \frac{74}{27} + \frac{27}{27}\right) = 1$$

$$z \cdot \left(\frac{117}{27}\right) = 1 \qquad z = \underline{\frac{27}{117}}$$

$$x = \frac{16}{27} \cdot \frac{27}{117} = \underline{\underline{\frac{16}{117}}} \qquad y = \frac{74}{27} \cdot \frac{27}{117} = \underline{\underline{\frac{74}{117}}} \qquad z = \underline{\underline{\frac{27}{117}}}$$

$$\simeq 0{,}137 \qquad \simeq 0{,}632 \qquad \simeq 0{,}231$$

Problem:

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

i) Compute determinant and rank

ii) Compute eigenvalues

iii) Check if A is diagonalizable

iv) Write down the corresponding quadratic form and determine its definiteness.

0) Solve $A \cdot \underline{x} = \underline{0}$ and $A \cdot \underline{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

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0)

$$\left(\begin{array}{cccc|cc} 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|cc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|cc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|cc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 3 & 0 & 1 \end{array}\right)$$

$x = 0$, $y = 0$, $z = 0$, $w = 0$

$x = 1/3$, $y = 1/3$, $z = 1/3$, $w = 1/3$

1) $rk\, A = \underline{4}$ $\quad \det A = -1 \cdot 1 \cdot 1 \cdot 3 = \underline{-3}$

i) <u>Det.</u> (alternative)

$$\begin{vmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{vmatrix} = +1 \cdot \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$+1 \cdot \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \left(1 \cdot 0 + 1 \cdot (-1)\right) - \left(1 \cdot 0 + 1 \cdot 1\right) + \left(1 \cdot (-1) - 1 \cdot 0\right)$$

$$= -1 - 1 - 1 = \underline{\underline{-3}}$$

ii) <u>Eigen values:</u>

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix}$$

$$\begin{vmatrix} 1-\lambda & 1 & 1 & 0 \\ 1 & 1-\lambda & 0 & 1 \\ 1 & 0 & 1-\lambda & 1 \\ 0 & 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 1 & 0 \\ 1-\lambda & 0 & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix}$$

$$(1-\lambda) \cdot \left[(1-\lambda) \cdot \left((1-\lambda)^2 - 1 \right) + 1 \cdot (-1)(1-\lambda) \right]$$

$$-1 \cdot \left[1 \cdot \left((1-\lambda)^2 - 1 \right) + 1 \cdot 1 \right]$$

$$+1 \cdot \left[1 \cdot (-1) - 1 \cdot \left((1-\lambda)^2 - 1 \right) \right]$$

$$= (1-\lambda)^4 + \Big(-(1-\lambda)^2 - (1-\lambda)^2 - (1-\lambda)^2 \cancel{+1} \cancel{-1} \cancel{-1}$$

$$- (1-\lambda)^2 \cancel{+1} \Big)$$

$$= (1-\lambda)^4 - 4\cdot(1-\lambda)^2 = 0$$

$$(1-\lambda)^2\cdot\left[(1-\lambda)^2 - 4\right] = 0$$

$$(1-\lambda)^2\cdot(\lambda^2 - 2\lambda - 3) = 0$$

$\underline{\lambda_1 = \lambda_2 = 1}$ $\qquad \lambda = \dfrac{2 \pm \sqrt{4 - 4\cdot(-3)}}{2} = \dfrac{2 \pm 4}{2}$

$\lambda_3 = \underline{3}$, $\lambda_4 = \underline{-1}$

iii) Is A diagonalizable? $\quad A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$

A is symmetric since $A^t = A$, therefore it is diagonalizable.

iv) Determine the definiteness of A.

$$x^2 + y^2 + z^2 + w^2 + 2xy + 2xz + 2yw + 2zw$$

$D_1 = \underline{1}$

$D_2 = \underline{0}$

$D_3 = 1(0-1) + 1\cdot(1-1) = \underline{-1}$

$\Bigg\}$ indefinite

Exam	Final exam in GRA 6035 Mathematics
Date	December 11th, 2015 at 0900 - 1200

All subquestions have the same weight and give maximal score 6p each. Answers to the first 12 subquestions give a maximal score of 72p (100%). Question 4 (c) can be skipped, but gives 6p extra credit if answered correctly.

Question 1.

We consider the matrix A given by

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

(a) **(6p)** Compute the rank of A and find all the solutions of the linear system $A \cdot \mathbf{x} = \mathbf{0}$.
(b) **(6p)** Is A diagonalizable? Find all eigenvalues of A and their multiplicities.
(c) **(6p)** Compute the eigenvalues of $B = A^2$. Is B diagonalizable?

We consider a Markov chain given by $\mathbf{x}_{t+1} = T\mathbf{x}_t$, where the transition matrix T is given by

$$T = \begin{pmatrix} 0.55 & 0.10 & 0.15 \\ 0.10 & 0.80 & 0.05 \\ 0.35 & 0.10 & 0.80 \end{pmatrix}$$

and the inital state is $\mathbf{x}_0$.

(d) **(6p)** Find the equilibrium state $\mathbf{x} = \lim_{t \to \infty} T^t \mathbf{x}_0$

Question 2.

Solve the differential equations:

(a) **(6p)** $y'' - 4y' - 12y = 15e^t$
(b) **(6p)** $y' = 3\sqrt{t} \cdot e^{-2y}$
(c) **(6p)** $4yt + 4t^3 + 2t + (2y - 1 + 2t^2)y' = 0, \quad y(1) = 0$

Question 3.

We consider the function given by $f(x, y, z) = \ln(u + 1)$, where $u = u(x, y, z)$ is the quadratic form

$$u(x, y, z) = 2x^2 + 2xy + 3y^2 - 2xz + z^2$$

(a) **(6p)** Determine the definiteness of u. Are there any points (x, y, z) where f is not defined?
(b) **(6p)** Compute the partial derivatives of f, and find all stationary points of f.
(c) **(6p)** Find the minimum value of f, if it exists. Is f convex?