PLENARY SESSION 4

GRA 6035 BI

Plan:

Problems: 7.4/8.3, 8.10, 8.11, 9.5
10.12, 10.14, 10.15, 11.11, 11.5, 11.6
Final exam 12/2013, Pb. 3-4.

7.4/8.3. max xyz when $\begin{cases} x^2+y^2=1 \\ x+z=1 \end{cases}$

$$\mathcal{L} = xyz - \lambda_1\cdot(x^2+y^2) - \lambda_2\cdot(x+z)$$

$$\boxed{\begin{aligned} \mathcal{L}'_x &= yz - \lambda_1\cdot 2x - \lambda_2 = 0 \\ \mathcal{L}'_y &= xz - \lambda_1\cdot 2y = 0 \\ \mathcal{L}'_z &= xy - \lambda_2 = 0 \\ & x^2 + y^2 = 1 \\ & x + z = 1 \end{aligned}}$$

$\Rightarrow yz - 2\lambda_1 x - xy = 0$

$\Rightarrow 2\lambda_1 = \frac{xz}{y}$

$\Rightarrow \lambda_2 = xy$

$y^2 = 1 - x^2$

$\Rightarrow z = 1 - x$

$$2\lambda_1 x = yz - xy$$

$$2\lambda_1 = \frac{yz}{x} - y = \frac{xz}{y} \quad \Big| \cdot xy$$

$$y^2z - xy^2 = x^2z$$

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$$(1-x^2)(1-x) - x(1-x^2) = x^2(1-x)$$

$(1-x)(1+x)$

$$(1-x^2)(1-x-x) = x^2(1-x)$$

$$(1-x^2)(1-2x) - x^2\underline{(1-x)} = 0$$

$$(1-x)\cdot\left[\underbrace{(1+x)(1-2x) - x^2}\right] = 0$$

$x=1$ or $-3x^2-x+1=0$

$$x = \frac{1 \pm \sqrt{1+12}}{-6} = \frac{-1 \pm \sqrt{13}}{6}$$

$\underline{x_1 = 1}$ $\quad \underline{x_2 = \frac{-1+\sqrt{13}}{6}}$ $\quad \underline{x_3 = \frac{-1-\sqrt{13}}{6}}$

$y=0$

$$y^2 = 1-x^2 \Rightarrow y = \pm\sqrt{1-x^2}$$

$$z = 1-x$$

$$\lambda_1 = \frac{1}{2}\left(\frac{yz}{x} - y\right) = \frac{y}{2}\left(\frac{z-x}{x}\right)$$

$$\lambda_2 = xy$$

$\Downarrow$

Five solutions: $(1,0,0;0,0)$ $\quad f=0$

$\left(x=\frac{-1+\sqrt{13}}{6}, \pm, \ldots, \ldots\right)$

$\left(x=\frac{-1-\sqrt{13}}{6}, \pm, \ldots\right)$

Solve max xyz s.t. $\begin{cases} x^2+y^2=1 \\ x+z=1 \end{cases}$

For each of the four last candidate pts compute f. $\to$ $(-0.77, -0.64, 1.77)$
$\shortparallel$
(x, y, z)

Yes, this is max

SOC: $\mathcal{L} = xyz - \lambda_1^*(x^2+y^2) - \lambda_2^*(x+z)$

$$H(\mathcal{L}) = \left(\begin{array}{cc|c} -2\lambda_1^* & z & y \\ z & -2\lambda_1^* & x \\ \hline y & x & 0 \end{array}\right)$$

$D_2 = 4(\lambda_1^*)^2 - z^2$

not concave

EVT: Is the adm. set bounded?

$$\left.\begin{array}{l} x^2+y^2=1 \leftarrow \begin{cases} -1 \le x \le 1 \\ -1 \le y \le 1 \end{cases} \\ x+z=1 \\ \Downarrow \\ z=1-x \to 0 \le z \le 2 \end{array}\right\} \Rightarrow \text{bounded}$$

$\Downarrow$

EVT holds

there is a maximum

NDCQ: $\operatorname{rk}\begin{pmatrix} 2x & 2y & 0 \\ 1 & 0 & 1 \end{pmatrix} = 2$

not any adm pts where NDCQ fails

fails $\Leftrightarrow$ $\operatorname{rk}\begin{pmatrix} 2x & 2y & 0 \\ 1 & 0 & 1 \end{pmatrix} < 2$

$-2y=0, \; 2y=0, \; 2x=0 \Leftrightarrow$ ~~$(x,y)=(0,0)$~~

$(x,y,z)=(0,0,z)$

$x^2+y^2 \neq 1$

not adm.

8.10. min $\underbrace{x^2+y^2+z^2}_{f}$ when $2x^2+6y^2+3z^2 \geq 36$

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KT: max $-f = -x^2-y^2-z^2$ when $-2x^2-6y^2-3z^2 \leq -36$

$$\mathcal{L} = -x^2-y^2-z^2-\lambda(-2x^2-6y^2-3z^2)$$
$$= -x^2-y^2-z^2+2\lambda x^2+6\lambda y^2+3\lambda z^2$$

FOC:
$$\begin{cases} \mathcal{L}'_x = -2x+4\lambda x = 0 \\ \mathcal{L}'_y = -2y+12\lambda y = 0 \\ \mathcal{L}'_z = -2z+6\lambda z = 0 \end{cases}$$

$$\boxed{\begin{aligned} x(-2+4\lambda) &= 0 \\ y(-2+12\lambda) &= 0 \\ z(-2+6\lambda) &= 0 \end{aligned}}$$

C: $2x^2+6y^2+3z^2 \geq 36$

CSC: $\lambda \geq 0$ and $\lambda \cdot (2x^2+6y^2+3z^2-36) = 0$

FOC:
$$\left.\begin{aligned} x=0 &\text{ or } \lambda = 1/2 \\ y=0 &\text{ or } \lambda = 1/6 \\ z=0 &\text{ or } \lambda = 1/3 \end{aligned}\right\}$$

$(x,y,z) = (0,0,0)$

not admissible since C gives $0 \geq 36$

$(x,y,z) = (0,0,z)$, $\lambda = 1/3$

$$\left.\begin{aligned} 3z^2 &\geq 36 \\ z^2 &\geq 12 \\ z \geq \sqrt{12} &\text{ or } z \leq -\sqrt{12} \end{aligned}\right\} (0,0,\pm\sqrt{12}; 1/3)\ f=12$$

$x=0$, $y,z \neq 0$

not possible

$$(x,y,z) = (0,y,0), \quad \lambda = 1/6$$

$$6y^2 = 36 \quad y^2 = 6 \quad y = \pm\sqrt{6}$$

$$(0, \pm\sqrt{6}, 0; 1/6)$$

$$f = 6$$

$$(x,0,0) \quad \lambda = 1/2$$

$$2x^2 = 36 \quad x^2 = 18 \Rightarrow x = \pm\sqrt{18}$$

$$(\pm\sqrt{18}, 0, 0; \tfrac{1}{2})$$

$$f = 18$$

Conclusion: Candidates

$(0,0,\pm\sqrt{12})$ $f=12$ $\lambda = 1/3$

$(0,\pm\sqrt{6},0)$ $f=6$ $\lambda = 1/6$ ← Best candidate

$(\pm\sqrt{18},0,0)$ $f=18$ $\lambda = 1/2$

SOC:

$$\mathcal{L} = -x^2 - y^2 - z^2 + 2\lambda^* x^2 + 6\lambda^* y^2 + 3\lambda^* z^2$$

$$= -x^2 - y^2 - z^2 + \tfrac{1}{3}x^2 + 1 \cdot y^2 + \tfrac{1}{2}z^2$$

$$= -\tfrac{2}{3}x^2 + 0 \cdot y^2 - \tfrac{1}{2}z^2$$

Concave

$\Downarrow$

$$H(\mathcal{L}) = \begin{pmatrix} -4/3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$(0,\pm\sqrt{6},0)$ is max for $-f$

$=$ min for f

with $f = 6$

Exam 12/2013, Q4

$$\max f = xw - yz \quad \text{when} \quad \begin{matrix} x^2+y^2 \le 1 \\ 4z^2+9w^2 \le 36 \end{matrix}$$

a) KT problem in std. form

$$\mathcal{L} = xw - yz - \lambda_1 \cdot (x^2+y^2) - \lambda_2(4z^2+9w^2)$$

KT cond.

FOC:

$$\begin{cases} \mathcal{L}'_x = w - \lambda_1 \cdot 2x = 0 \\ \mathcal{L}'_y = -z - \lambda_1 \cdot 2y = 0 \\ \mathcal{L}'_z = -y - \lambda_2 \cdot 8z = 0 \\ \mathcal{L}'_w = x - \lambda_2 \cdot 18w = 0 \end{cases}$$

C:

$$\begin{cases} x^2+y^2 \le 1 \\ 4z^2+9w^2 \le 36 \end{cases}$$

CSC:

$$\begin{cases} \lambda_1 \ge 0 \text{ and } \lambda_1 \cdot (x^2+y^2-1) = 0 \\ \lambda_2 \ge 0 \text{ and } \lambda_2 \cdot (4z^2+9w^2-36) = 0 \end{cases}$$

$(x,y,z,w) = (0,1,-3,0)$:

$0 = 0$ ok.

$3 - 2\lambda_1 = 0$ $\quad \lambda_1 = 3/2$ ok.

$-1 + 24\lambda_2 = 0$ $\quad \lambda_2 = 1/24$ ok.

$0 = 0$ ok.

$0 + 1 \le 1$ ok.

$36 + 0 \le 36$ ok.

$\lambda_1 = 3/2, \lambda_2 = 1/24 > 0$

ok. $\frac{3}{2} \cdot (1-1) = 0$ ✓

ok. $\frac{1}{24}(36-36) = 0$ ✓

$\hookrightarrow$ $(x,y,z,w) = (0,1,-3,0)$ is a solution to the KT cond. with $\lambda_1 = 3/2$, $\lambda_2 = 1/24$

b) Candidate: $(x, y, z, w; \lambda_1, \lambda_2)$

$= (0, 1, -3, 0; 3/2, 1/24)$

Use SOC: $\mathcal{L} = xw - yz - \frac{3}{2}(x^2 + y^2) - \frac{1}{24}(4z^2 + 9w^2)$

$$H(\mathcal{L}) = \begin{pmatrix} -3 & 0 & 0 & 1 \\ 0 & -3 & -1 & 0 \\ 0 & -1 & -1/3 & 0 \\ 1 & 0 & 0 & -3/4 \end{pmatrix}$$

Leading principle minors:

$D_1 = -3$

$D_2 = 9$

$D_3 = -3 \cdot (1 - 1) = 0$

$D_4 = -1 \cdot 1 \cdot 0 - 3/4 \cdot 0$
$= 0$

Other principal minors:

$\Delta_1 = -3, -1/3, -3/4$

$\Delta_2 = 1, 5/4, 0, 9/4, 1/4$

$\Delta_3 = 0, -15/4, -5/12$

$$\Delta_1 \leq 0, \quad \Delta_2 \geq 0, \quad \Delta_3 \leq 0, \quad \Delta_4 \geq 0$$

$\Downarrow$

$\mathcal{L}$ concave

$\Downarrow$

$(0, 1, -3, 0)$ is max

c) KT problem with parameter:

$$\max\ f = xw - yz \quad \text{when} \quad \begin{cases} x^2+y^2 \le 1 \\ az^2 + 9w^2 \le 36 \end{cases}$$

parameter

$\underline{a=4}$: $(x,y,z,w) = (0,1,-3,0)$ with $\lambda_1 = 3/2$, $\lambda_2 = 1/24$, $f=3$ $\Big\}$ $f^*(4) = 3$

Use Envelope thm to estimate $f^*(4.2)$.

Env. thm: $\dfrac{df^*(a)}{da} = \dfrac{\partial \mathcal{L}}{\partial a}\Big|_{\underline{x}=x^*(a),\ \underline{\lambda}=\lambda^*(a)}$

$$\mathcal{L} = xw - yz - \lambda_1(x^2+y^2) - \lambda_2\cdot(az^2+9w^2)$$

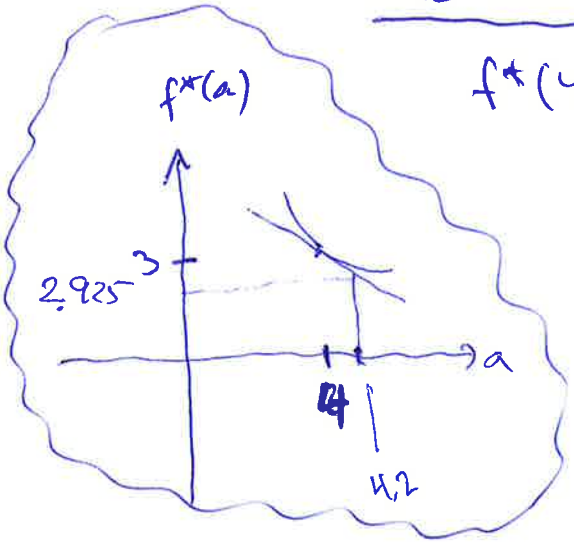
$$\frac{\partial \mathcal{L}}{\partial a} = -\lambda_2 z^2$$

$$\frac{df^*(a)}{da} = -\lambda_2^* \cdot (z^*)^2 = -\lambda_2^*(a)\left(z^*(a)\right)^2$$

$$= -\frac{1}{24}\cdot(-3)^2 = -\frac{9}{24}$$

at $a=4$

This means: Linear approximation



$$f^*(4.2) \simeq f^*(4) + (4.2-4)\cdot \underbrace{\frac{df^*(a)}{da}}_{=-9/24}$$

$$= 3 + 0.2\cdot(-9/24) = 3 - \frac{9}{120}$$

$$= 3 - \frac{3}{40} = 3 - 0{,}075 = \underline{\underline{2.925}}$$

10.12.

a) $ty' + 2y + t = 0 \quad (t \neq 0)$

$y' + \frac{2}{t}y + 1 = 0$

$y' + \frac{2}{t}y = -1 \quad | \cdot t^2$

$a(t) = \frac{2}{t} \quad b(t) = -1$

Int.-factor:

$\int a(t)\,dt = \int \frac{2}{t}\,dt$

$= 2\ln|t| + c$

$e^{2\ln t} = e^{\ln t \cdot 2}$

$= (t)^2 = t^2$

$t^2 y' + 2ty = -t^2$

$(t^2 \cdot y)' = -t^2$

$t^2 \cdot y = \int -t^2\,dt = -\frac{1}{3}t^3 + c$

$y = -\frac{1}{3}t + \frac{c}{t^2} \quad (t \neq 0)$

10.14. $2t + 3y^2 \cdot y' = 0$

Seperable: $3y^2 y' = -2t$

$y' = \frac{-2t}{3y^2} = \frac{1}{3y^2} \cdot (-2t) \quad | \cdot 3y^2$

$3y^2 \cdot y' = -2t$

$\int 3y^2\,dy = \int -2t\,dt$

$y^3 = -t^2 + C$

$y = \sqrt[3]{C - t^2}$

Exact:

$$2t + 3y^2 \cdot y' = 0$$

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial y} \cdot \frac{dy}{dt} = 0$$

$$\left.\begin{aligned} h'_t &= 2t \\ h'_y &= 3y^2 \end{aligned}\right\} \quad \begin{aligned} h &= t^2 + C(y) \\ h' &= 0 + C'(y) = 3y^2 \\ C'(y) &= 3y^2 \\ C(y) &= y^3 . \end{aligned}$$

$$h = t^2 + y^3 = C$$

$$y^3 = C - t^2$$

$$y = \underline{\underline{\sqrt[3]{C - t^2}}}$$

10.15. $(2t+y)-(4y-t)y'=0 \quad , y(0)=0$

Separable: $\cancel{y'=\frac{2t+y}{4y-t}}$

Exact: $\frac{\partial h}{\partial t}+\frac{\partial h}{\partial y}\cdot\frac{dy}{dt}=0$

$$\boxed{\begin{aligned} h'_t &= 2t+y \\ h'_y &= -(4y-t)=t-4y \end{aligned}}$$

$h = t^2+yt+C(y)$

$h'_y = \cancel{0}+t+C'(y) = t-4y$

$C'(y) = -4y$

$C(y) = -2y^2$

$h = \boxed{t^2+yt-2y^2=C}$

$y(0)=0$:

$t=0, y=0$

$0+0-0=C \Rightarrow C=0$

$t^2+yt-2y^2=0$

$(-2)y^2+(t)\cdot y+(t^2)=0$

$y=\frac{-t\pm\sqrt{t^2-4\cdot(-2)(t^2)}}{-4}$

$=\frac{-t\pm\sqrt{9t^2}}{-4}=\frac{-t\pm 3t}{-4}$

$y=-\frac{t}{2}$

or

$y=t$

11.5. $y'' + ay' + by = 0$

$r^2 + ar + b = 0$

$r = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$

$a^2 - 4b = 0$
$\Updownarrow$
$r = -a/2$ double root

Let $y = u \cdot e^{rt}$ $(r = -a/2)$

$y' = u' \cdot e^{rt} + u \cdot e^{rt} \cdot r = (u' + ru)e^{rt}$

$y'' = (u'' + ru')e^{rt} + (u' + ru) \cdot e^{rt} \cdot r$

$= (u'' + ru' + ru' + r^2 u) e^{rt}$

$= (u'' + 2ru' + r^2 u) e^{rt}$

$y'' + ay' + by = 0$

$\frac{1}{e^{rt}}$ $(u'' + 2ru' + r^2 u)e^{rt} + a(u' + ru)e^{rt} + b(ue^{rt}) = 0$

$u'' + 2ru' + r^2 u + au' + aru + bu = 0$

$u'' + (2r + a)u' + (r^2 u + aru + bu) = 0$

$r = -a/2 \rightarrow$ $u'' + \underbrace{(\cancel{2} \cdot (-a/\cancel{2}) + a)}_{0} u' + \underbrace{\left(\frac{a^2}{4} - \frac{a^2}{2} + b\right)}_{-\frac{a^2}{4} + b = 0} u = 0$

$a^2 = 4b$:

$-\frac{\cancel{4}b}{\cancel{4}} + b$

$\boxed{u'' = 0}$ $u' = C$ $u = Ct + D$

Solutions: $y = (Ct + D) \cdot e^{rt}$ is a solution

11.6.

Sec. order lin. nonhomogeneous

a) $x'' - x = e^{-t}$

$$X = X_h + X_p = C_1 e^{t} + C_2 e^{-t} - \frac{1}{2} t e^{-t}$$

X_h: $x'' - x = 0$

Char. eqn: $r^2 - 1 = 0$

$r = \pm 1 \Rightarrow X_h = C_1 e^{t} + C_2 e^{-t}$

X_p: $x'' - x = e^{-t} \longrightarrow f(t) = e^{-t}$

$f'(t) = e^{-t} \cdot (-1) = -e^{-t}$

$f''(t) = -e^{-t} \cdot (-1) = e^{-t}$

Guess: $\cancel{y} = Ae^{-t}$

$\cancel{y}' = Ae^{-t} \cdot (-1) = -Ae^{-t}$

$\cancel{y}'' = Ae^{-t}$

$Ae^{-t} - Ae^{-t} = e^{-t}$

$(A - A)e^{-t} = e^{-t}$ no solution

New Guess: $X = t \cdot Ae^{-t} = Ate^{-t}$

$x' = A \cdot e^{-t} + Ate^{-t} \cdot (-1)$

$= (A - At)e^{-t}$

$x'' = -A \cdot e^{-t} + (A - At) e^{-t} (-1)$

$= (At - 2A)e^{-t}$

$$x'' - x = e^{-t}$$

$$(At - 2A)e^{-t} - Ate^{-t} = e^{-t}$$

$$(\cancel{At} - 2A - \cancel{At})e^{-t} = e^{-t}$$

$$-2A \cdot e^{-t} = e^{-t}$$

$$-2A = 1 \qquad A = \underline{-1/2} \qquad x_p = \underline{-\tfrac{1}{2}te^{-t}}$$