PLENARY SESSION 3

GRA 6035 BI

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MATHEMATICS

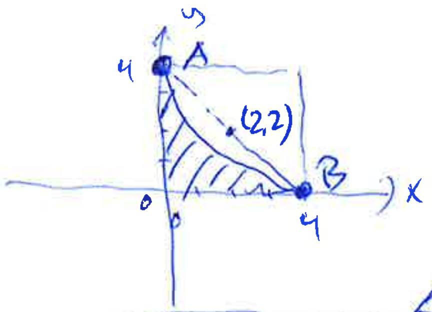
Problems:

Lecture 7/8/9: 7.1, 7.5, 7.7/8.6/9.4, 7.8/8.7, 7.9/8.8, 7.10/8.9, 8.12, 8.13, ~~[illegible]~~, 9.9

7.1 b) $\sqrt{x}+\sqrt{y} \leq 2$ $D = \{(x,y) : \sqrt{x}+\sqrt{y} \leq 2\}$

open/closed? closed $\leq$ (not open)

bounded? Yes



$x \geq 0$ $x \leq 4$
$y \geq 0$ $y \leq 4$

$\sqrt{y} \leq 2 - \sqrt{x}$

Boundary: $\sqrt{x}+\sqrt{y} = 2$

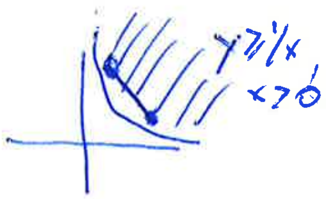
$\sqrt{y} = 2 - \sqrt{x}$

$y = (2-\sqrt{x})^2$

$= 4 - 4\sqrt{x} + x$

$x^2+y^2 \geq 8$

not convex



convex: A set D is convex if for any pts A, B in D, the line segment [A, B] is inside D

$A = (0,4)$
$B = (4,0)$ $\Big\}$ $(2,2) \in [A,B]$ $\sqrt{x}+\sqrt{y} = \sqrt{2}+\sqrt{2} > 2$

not convex set not in D

78. max xy when $x^2+y^2 \le 1$ ← KT, std. form. BI

$$\mathcal{L} = xy - \lambda\cdot(x^2+y^2)$$

FOC
$$\mathcal{L}'_x = y - \lambda\cdot 2x = 0$$
$$\mathcal{L}'_y = x - \lambda\cdot 2y = 0$$

C
$$x^2+y^2 \le 1$$

CSC
$$\lambda \ge 0 \quad \text{and} \quad \lambda\cdot(x^2+y^2-1) = 0$$

$y = \underline{2\lambda\cdot x}$

$x = 2\lambda\cdot y = 2\lambda\cdot(2\lambda\cdot x) = 4\lambda^2 x$

$x - 4\lambda^2 x = 0$

$x\cdot(1-4\lambda^2) = 0$

$\underline{x=0}$ or $\underline{1-4\lambda^2=0}$

$x=0$, $y=0$, $\lambda=0$ | $\lambda^2 = 1/4$

$f=0$ $\quad \cancel{(0,0;0)}$ | $\lambda = \pm 1/2 = 1/2$

$y = x$, $2x^2 \le 1$

$2x^2 = 1 \quad x^2 = 1/2 \quad x = \pm\sqrt{1/2}$

$(\sqrt{1/2}, \sqrt{1/2}; 1/2) \quad f = 1/2$

$(-\sqrt{1/2}, -\sqrt{1/2}; 1/2) \quad f = 1/2$

3 candidates from KT-cond.

these are maximizers and $f = 1/2$ is max value

8.7 Try SOC:

$$\mathcal{L} = xy - \lambda^*\cdot(x^2+y^2) = xy - \frac{1}{2}(x^2+y^2)$$

$$\left.\begin{aligned}\mathcal{L}'_x &= y - x\\ \mathcal{L}'_y &= x - y\end{aligned}\right\} \quad H(\mathcal{L}) = \begin{pmatrix} \boxed{-1} & 1 \\ 1 & -1 \end{pmatrix} \quad \begin{aligned} D_1 &= -1 \\ D_2 &= 0 \end{aligned}$$

$\Delta_1 = -1, -1 \le 0$

$\Delta_2 = 0 \ge 0$

concave

7.7. max $x+4y+z$ when $\begin{cases} x^2+y^2+z^2=216 \\ x+2y+3z=0 \end{cases}$ ← Lagrange pb.

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$$\mathcal{L} = x+4y+z-\lambda_1\cdot(x^2+y^2+z^2)-\lambda_2\cdot(x+2y+3z)$$

FOC $\begin{cases} \mathcal{L}'_x = 1-\lambda_1 2x-\lambda_2=0 \\ \mathcal{L}'_y = 4-\lambda_1\cdot 2y-2\lambda_2=0 \\ \mathcal{L}'_z = 1-\lambda_1\cdot 2z-3\lambda_2=0 \end{cases}$

C $\quad x^2+y^2+z^2=216$

$\quad x+2y+3z=0$

$2\lambda_1 x = 1-\lambda_2 \qquad x=\frac{1-\lambda_2}{2\lambda_1}$

$2\lambda_1 y = 4-2\lambda_2 \qquad y=\frac{4-2\lambda_2}{2\lambda_1}$

$2\lambda_1 z = 1-3\lambda_2 \qquad z=\frac{1-3\lambda_2}{2\lambda_1}$

$$x+2y+3z=0$$

$$\frac{1-\lambda_2}{2\lambda_1}+2\cdot\frac{4-2\lambda_2}{2\lambda_1}+3\cdot\frac{1-3\lambda_2}{2\lambda_1}=0 \quad \Big|\, 2\lambda_1$$

$$(1-\lambda_2)+2\cdot(4-2\lambda_2)+3\cdot(1-3\lambda_2)=0$$

$$12-14\lambda_2=0 \qquad \lambda_2=\frac{12}{14}=\frac{6}{7}$$

$$x=\frac{1-6/7}{2\lambda_1}=\frac{1/7}{2\lambda_1}=\frac{1}{14\lambda_1}$$

$$y=\frac{4-2\cdot 6/7}{2\lambda_1}=\frac{16}{14\lambda_1}$$

$$z=\frac{1-3\lambda_2^{6/7}}{2\lambda_1}=\frac{-11}{14\lambda_1}$$

$$x^2+y^2+z^2=216$$

$$\frac{1}{(14\lambda_1)^2}+\frac{16^2}{(14\lambda_1)^2}+\frac{(-11)^2}{(14\lambda_1)^2}=216$$

$$\frac{1+16^2+11^2}{14^2\lambda_1^2}=216$$

$$\frac{378}{14^2\lambda_1^2}=216$$

$$\lambda_1^2=\frac{378}{216\cdot 14^2}$$

$$\lambda_1=\pm\sqrt{\frac{378}{216}}\cdot\frac{1}{14}=\pm\sqrt{\frac{7}{4}}\cdot\frac{1}{14}=\pm\frac{\sqrt{7}}{28}$$

$$14\lambda_1=\pm\frac{\sqrt{7}}{2}$$

$378=3\cdot 126=3\cdot 2\cdot 63=3\cdot 2\cdot 9\cdot 7$

$216=2\cdot 108=2\cdot 3\cdot 36=2\cdot 3\cdot 9\cdot 4$

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$$x = \pm \frac{2}{\sqrt{7}} \qquad y = \pm \frac{2}{\sqrt{7}} \cdot 16 = \pm \frac{32}{\sqrt{7}} \qquad z = \mp \frac{22}{\sqrt{7}}$$

$$\lambda_1 = \pm \frac{\sqrt{7}}{28} \qquad \lambda_2 = \frac{6}{7}$$

$$(x, y, z; \lambda_1, \lambda_2) = \left(\frac{2}{\sqrt{7}}, \frac{32}{\sqrt{7}}, -\frac{22}{\sqrt{7}}; \frac{\sqrt{7}}{28}, \frac{6}{7}\right) \qquad f = \frac{108}{\sqrt{7}}$$

$$= \left(-\frac{2}{\sqrt{7}}, -\frac{32}{\sqrt{7}}, \frac{22}{\sqrt{7}}; -\frac{\sqrt{7}}{28}, \frac{6}{7}\right) \qquad f = \frac{-108}{\sqrt{7}}$$

$f = x + 4y + z$: Cand. for max

8.6: Try SOC: $\lambda_1^* = \frac{\sqrt{7}}{28}$ $\lambda_2^* = \frac{6}{7}$

$$\mathcal{L} = x + 4y + z - \lambda_1^*(x^2 + y^2 + z^2) - \lambda_2^* \cdot (x + 2y + 3z)$$

$$H(\mathcal{L}) = \begin{pmatrix} -2 \cdot \sqrt{7}/28 & 0 & 0 \\ 0 & -2 \cdot \sqrt{7}/28 & 0 \\ 0 & 0 & -2 \cdot \sqrt{7}/28 \end{pmatrix} \qquad \begin{aligned} D_1 &= -\frac{\sqrt{7}}{14} < 0 \\ D_2 &= +7/14^2 > 0 \\ D_3 &= -\frac{7\sqrt{7}}{14 \cdot 14^2} < 0 \end{aligned}$$

Concave $\Leftarrow$ neg. defn.

$\Downarrow$

$\left(\frac{2}{\sqrt{7}}, \frac{32}{\sqrt{7}}, -\frac{22}{\sqrt{7}}\right)$ is maximizer

$f = \frac{108}{\sqrt{7}}$ is max

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<u>94</u> <u>Envelope thm:</u>

$$\max \quad x+4y+z \quad \text{when} \quad \begin{cases} x^2+y^2+z^2=a \\ x+2y+3z=b \end{cases}$$

(Lagrange pb. with parameters a, b)

$$\mathcal{L} = x+4y+z-\lambda_1\cdot(x^2+y^2+z^2-a)-\lambda_2\cdot(x+2y+3z-b)$$

<u>$a=216,\ b=0$:</u> $\left(\frac{2}{\sqrt7}, \frac{32}{\sqrt7}, \frac{-22}{\sqrt7}; \frac{\sqrt7}{28}, \frac{6}{7}\right)$ $\quad f=\frac{108}{\sqrt7}$

$$f^*(216,0) = 108/\sqrt7$$

<u>Envelope thm:</u>

$$\frac{\partial f^*}{\partial a} = \frac{\partial \mathcal{L}}{\partial a}\bigg|_{x^*,\lambda^*} = \lambda_1^* = \frac{\sqrt7}{28}$$

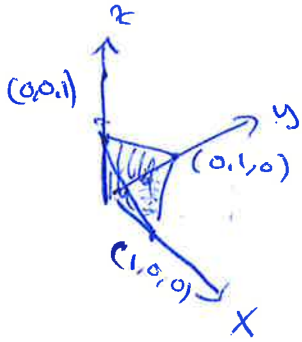
$$\frac{\partial f^*}{\partial b} = \frac{\partial L}{\partial b}\bigg|_{x^*,\lambda^*} = \lambda_2^* = \frac{6}{7}$$

$$f^*(215,0.1) \simeq f^*(216,0) + \underbrace{(-1)}_{\Delta a \,=\, 215-216}\cdot\underbrace{\frac{\sqrt7}{28}}_{=\frac{\partial f^*}{\partial a}} + \overbrace{0.1}^{0.1-0 \,=\, \Delta b}\cdot\underbrace{\frac{6}{7}}_{=\frac{\partial f^*}{\partial b}}$$

$$= \frac{108}{\sqrt7} - \frac{\sqrt7}{28} + \frac{0.6}{7}$$

$$\simeq 40.820 - 0.009 = \underline{\underline{40.811}}$$

7.9 max xyz when $\begin{cases} x \geq 0 \\ y \geq 0 \\ z \geq 0 \\ x+y+z \leq 1 \end{cases}$



$= \max \; xyz$ when $\begin{cases} -x \leq 0 \\ -y \leq 0 \\ -z \leq 0 \\ x+y+z \leq 1 \end{cases}$ ← KT-pb. in std. form

$$\mathcal{L} = xyz - \mu_1(-x) - \mu_2 \cdot (-y) - \mu_3(-z) - \lambda \cdot (x+y+z)$$

$$\boxed{\begin{array}{lll}
\text{FOC} & \mathcal{L}'_x = yz + \mu_1 - \lambda = 0 & \\
& \mathcal{L}'_y = xz + \mu_2 - \lambda = 0 & \\
& \mathcal{L}'_z = xy + \mu_3 - \lambda = 0 & \\
\text{C} & x \geq 0 & \\
& y \geq 0 & \\
& z \geq 0 & \\
& x+y+z \leq 1 & \\
\text{CSC} & \mu_1 \geq 0 \quad \text{and} & \mu_1 x = 0 \\
& \mu_2 \geq 0 & \mu_2 y = 0 \\
& \mu_3 \geq 0 & \mu_3 z = 0 \\
& \lambda \geq 0 & \lambda \cdot (x+y+z-1) = 0
\end{array}}$$

a) $\lambda = 0$:

b) $\lambda > 0$: $x+y+z = 1$

a) $\underline{\lambda = 0}$: FOC

$\underline{\mu_1 = \mu_2 = \mu_3 = 0}$

$x=y=0$, or

$x=z=0$, or

$y=z=0$

$yz + \mu_1 = 0 \Rightarrow \mu_1 = 0$, $yz = 0$

$xz + \mu_2 = 0$ $\quad \mu_2 = 0$, $xz = 0$

$xy + \mu_3 = 0$ $\quad \mu_3 = 0$, $xy = 0$

We set solutions in a):

$\lambda=0,\ \mu_1=\mu_2=\mu_3=0$ $\left|\begin{array}{l} x=y=0,\ 0\le z\le 1 \\ \text{or} \\ x=z=0,\ 0\le y\le 1 \\ y=z=0,\ 0\le x\le 1 \end{array}\right.$

All these points have $f=xyz=\underline{0}$

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b) $\underline{\lambda>0}$: $x+y+z=1$

$\left.\begin{array}{l} \underline{x>0} \\ \underline{y>0} \\ \underline{z>0} \end{array}\right\}$ $\begin{array}{l} \mu_1=0 \\ \mu_2=0 \\ \mu_3=0 \end{array}$

(most interesting part)

FOC: $yz-\lambda=0$
$xz-\lambda=0$
$xy-\lambda=0$

$\lambda=\underbrace{xy=xz}_{y=z}=\underbrace{yz}_{x=y}$

$x=y=z$

$x+y+z=1$
$3x=1 \Rightarrow x=1/3$

$\boxed{x=1/3,\ y=1/3,\ z=1/3,\ \lambda=1/9,\ \mu_1=\mu_2=\mu_3=0}$

$f=xyz=\underline{1/27}$

$\lambda>0$, $x+y+z=1$, either $x=0$, or $y=0$ or $z=0$
But then $f=xyz=0<1/27$, $\underline{\underline{\text{not max}}}$

$\underline{\text{Conclusion}}$: $(x,y,z)=(1/3,1/3,1/3)$ is $\underline{\text{the best cand. for max}}$

Since xyz is not concave, SOC will not work here.

$$L = xyz + \mu_1 x + \mu_2 y + \mu_3 z - \lambda(x+y+z)$$

Proof that $(1/3, 1/3, 1/3)$ is max:

The set of admissible pts is bounded therefore there is a max by EVT.

$0 \leq x \leq 1$
$0 \leq y \leq 1$
$0 \leq z \leq 1$

Either 1) Ordinary: $(1/3, 1/3, 1/3)$ best cand.

2) Special: Adm pt where NDCQ fails. (tedious comp. but NDCQ holds at all adm. pts)

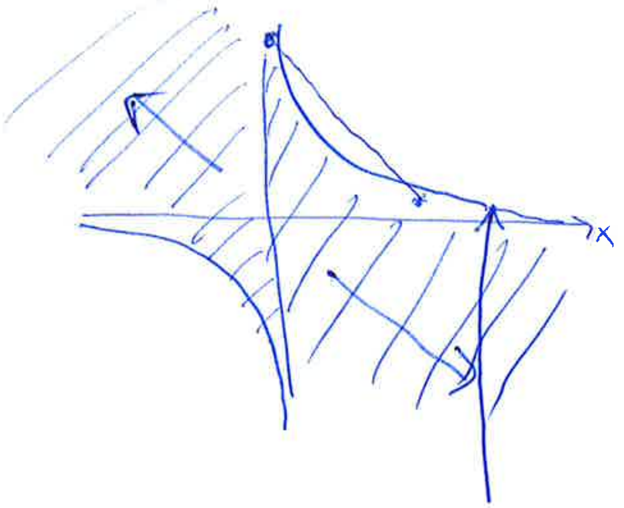
$\Downarrow$

$(1/3, 1/3, 1/3)$ is the max $f = 1/27$

7.1 $xy \le 1$

closed (not open) $\le$

BI



$y = f(x) = 1/x$ is convex function

$xy = 1$
$y = 1/x$

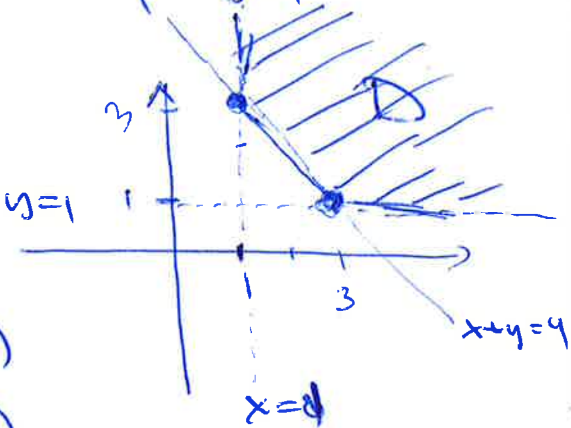
$\underline{x > 0}$: $xy \le 1$
$y \le 1/x$

$\underline{x < 0}$: $xy \le 1$
$y \ge 1/x$

not bounded

not convex

8.13. max $\ln(x^2 y) - x - y$ when $\begin{cases} x + y \ge 4 \\ x \ge 1 \\ y \ge 1 \end{cases}$



$-x - y \le -4$
$-x \le -1$
$-y \le -1$

$$\mathcal{L} = \ln(x^2 y) - x - y - \lambda_1(-x-y) - \lambda_2(-x) - \lambda_3(-y)$$

$$= 2\ln(x) + \ln(y) - x - y + \lambda_1(x+y) + \lambda_2 x + \lambda_3 y$$

FOC: $\begin{cases} \mathcal{L}'_x = \frac{2}{x} - 1 + \lambda_1 + \lambda_2 = 0 \\ \mathcal{L}'_y = \frac{1}{y} - 1 + \lambda_1 + \lambda_3 = 0 \end{cases}$

C: $\{ x + y \ge 4, \; x \ge 1, \; y \ge 1$

CSC: $\begin{cases} \lambda_1 \ge 0 & \lambda_1 \cdot (x + y - 4) = 0 \\ \lambda_2 \ge 0 & \lambda_2 \cdot (x - 1) = 0 \\ \lambda_3 \ge 0 & \lambda_3 \cdot (y - 1) = 0 \end{cases}$

a) $x = 1$: $1 + \lambda_1 + \lambda_2 = 0$ imp.

b) $y = 1$: $\lambda_1 + \lambda_3 = 0 \Rightarrow \lambda_1 = \lambda_3 = 0$

$\frac{2}{x} - 1 + \lambda_2 = 0$, $x \ge 3$, $\lambda_2 = 0$ $\Big\}$ $\frac{2}{x} - 1 = 0$, $x = 2$

imp.

c) $\underline{x+y=4}$: $x\neq 1, y\neq 1 \Rightarrow x>1, y>1 \Rightarrow \underline{\lambda_2=\lambda_3=0}$

$$\frac{2}{x}-1+\lambda_1=0 \qquad \lambda_1 = 1-\frac{2}{x} = 1-\frac{1}{y}$$

$$\frac{1}{y}-1+\lambda_1=0$$

$$\frac{2}{x}=\frac{1}{y}$$

$$2y=x$$

$$2y+y=4$$

$$3y=4$$

$$\underline{y=4/3}$$

$$\underline{x=8/3}$$

$\underline{\text{Cand:}}$ $\boxed{\begin{array}{l} x=8/3 \\ y=4/3 \\ \lambda_1=1/4 \\ \lambda_2=\lambda_3=0 \end{array}}$

$$f=\ln(x^2y)-x-y$$
$$=\ln\left(\tfrac{64}{9}\cdot\tfrac{4}{3}\right)-4 \simeq -1.75$$

this is $\underline{\text{the max}}$

(see the workbook)

d) $\underline{x>1, y>1, x+y>4}$: $\lambda_1=\lambda_2=\lambda_3=0$

$$\left.\begin{array}{l} \frac{2}{x}-1=0 \quad \underline{x=2} \\ \frac{1}{y}-1=0 \quad \underline{y=1} \quad \text{imp.} \end{array}\right\} \Rightarrow \text{no } \underline{\text{cand.}}$$

$\underline{\text{Only one cand}}$ from KT-cond. $x=8/3$ $y=4/3$ $\lambda_1=1/4$ $\lambda_2=\lambda_3=0$

$\underline{\text{SOC}}$: $L = 2\ln x+\ln y-x-y+\frac{1}{4}(x+y)$

$$H(L)=\begin{pmatrix} -2/x^2 & 0 \\ 0 & -1/y^2 \end{pmatrix} \qquad \begin{array}{l} D_1=-2/x^2<0 \\ D_2=2/x^2y^2>0 \end{array}$$

pos. defn.
$\Downarrow$
concave
$\Downarrow$

$$\underline{\underline{(x,y)=(8/3, 4/3)}}$$

is max