PLENARY SESSION 1

GRA 6035 BI

MATHEMATICS

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Plan:

Selected exercises from the Workbook, Lecture 1 - 3

Problems

1.4, 1.8-1.10, 1.15 (1.11)
2.18, 2.20c, 2.21
3.4, 3.5, 3.14, 3.15 (3.11, 3.12)

Lecture 1:

1.4

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 4 \\ 1 & 2 & 1 & h \end{array}\right) \xrightarrow{\cdot(-1)} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 3 \\ 0 & 1 & 0 & h-1 \end{array}\right) \cdot \tfrac{1}{2}$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 3 \\ 0 & 0 & 0 & h+1/2 \end{array}\right)$$

Consistent

$\Updownarrow$

$h + 1/2 = 0$

$\underline{\underline{h = -1/2}}$

1.15

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2q \\ 2 & -3 & 2 & 4q \\ 3 & -2 & p & q \end{array}\right) \begin{array}{c} \\ -2 \\ -3 \end{array} \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2q \\ 0 & -5 & 0 & 0 \\ 0 & -5 & p-3 & -5q \end{array}\right) \cdot(-1)$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2q \\ 0 & -5 & 0 & 0 \\ 0 & 0 & p-3 & -5q \end{array}\right)$$

$p = 3$:
- $q = 0$: inf. many solutions, $\text{rk}\, A = 2$, $\text{rk}\, \hat{A} = 2$ (x_3 free)
- $q \neq 0$: no solutions, $\text{rk}\, A = 2$, $\text{rk}\, \hat{A} = 3$

$p \neq 3$: One solution, $\text{rk}\, A = \text{rk}\, \hat{A} = 3$

1.8

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ -1 & a & -21 & 2 \\ 3 & 7 & a & b \end{array}\right) \cdot 1, \cdot(-3) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & a+2 & -18 & 3 \\ 0 & 1 & a-9 & b-3 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & a-9 & b-3 \\ 0 & a+2 & -18 & 3 \end{array}\right) \cdot(-(a+2))$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & a-9 & b-3 \\ 0 & 0 & \underbrace{-18-(a+2)(a-9)}_{*} & \underbrace{3-(a+2)(b-3)}_{**} \end{array}\right)$$

$\underline{* = 0:}$ $\quad -\cancel{18} - (a^2 - 7a - \cancel{18}) = 0$

$-a^2 + 7a = 0 \iff a^2 - 7a = 0$

$\underline{a = 0}, \quad \underline{a = 7}$

$\underline{** = 0:}$ $\quad 3 - (a+2)(b-3) = 0$

$a = 0: \quad 3 - 2b + 6 = 0 \quad 2b = 9 \quad \underline{b = 9/2}$

$a = 7: \quad 3 - 9b + 27 = 0 \quad 9b = 30 \quad b = 30/9 = \underline{10/3}$

$a = 0, 7$: $\quad ** = 0$ infinitely many solutions

$\quad ** \neq 0$ no solutions

$a \neq 0, 7$: $\quad$ <u>one solution</u>

<u>Conclusion:</u>

$a \neq 0, a \neq 7$: one solution

$a = 0, b \neq 9/2$: no solutions

$a = 7, b \neq 10/3$: no solution

$a = 0, b = 9/2$: inf. many

$a = 7, b = 10/3$: inf. —"—

1.9

$$\begin{pmatrix} 1 & 3 & 4 & 1 & 7 \\ 3 & 2 & 1 & 0 & 7 \\ -1 & 3 & 2 & 4 & 9 \end{pmatrix}$$

-3, 1

$$\begin{pmatrix} 1 & 3 & 4 & 1 & 7 \\ 0 & -7 & -11 & -3 & -14 \\ 0 & 6 & 6 & 5 & 16 \end{pmatrix}$$

$6/7$

$$\begin{pmatrix} 1 & 3 & 4 & \cdots & \cdots \\ 0 & -7 & -11 & \cdots & \cdots \\ 0 & 0 & -66/7 + 42/7 & \cdots & \cdots \end{pmatrix}$$

$-66/7 + 42/7 \neq 0$

1.10

$$\left(\begin{array}{cccc|c} 1 & 6 & -7 & 3 & 1 \\ 1 & 9 & -6 & 4 & 2 \\ 1 & 3 & -8 & 4 & 5 \end{array}\right)$$

-1, -1

$$\left(\begin{array}{cccc|c} 1 & 6 & -7 & 3 & 1 \\ 0 & 3 & 1 & 1 & 1 \\ 0 & -3 & -1 & 1 & 4 \end{array}\right)$$

1

$$\left(\begin{array}{cccc|c} 1 & 6 & -7 & 3 & 1 \\ 0 & 3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 5 \end{array}\right)$$

free

inf. many solutions, one free variable (z)

free variables $= n - \text{rk} A = 4 - 3 = \underline{1}$

1.11 a)

$$x - 3y + 6z = -1$$
$$2x - 5y + 10z = 0$$
$$3x - 8y + 17z = 1$$

i) Substitución:

$x = 3y - 6z - 1 \rightarrow 2(3y - 6z - 1) - 5y + 10z = 0$

$\boxed{y - 2z = 2}$

$\rightarrow 3(3y - 6z - 1) - 8y + 17z = 1$

$\boxed{y - z = 4}$

$$\boxed{\begin{matrix} y - 2z = 2 \\ y - z = 4 \end{matrix}}$$

$y = 2z + 2 \rightarrow (2z + 2) - z = 4$

$z = 2$

$y = 6$

$x = 5$

ii)
$$\left(\begin{array}{ccc|c} 1 & -3 & 6 & -1 \\ 2 & -5 & 10 & 0 \\ 3 & -8 & 17 & 1 \end{array}\right) \begin{matrix} -2 \\ -3 \end{matrix} \rightarrow \left(\begin{array}{ccc|c} 1 & -3 & 6 & -1 \\ 0 & 1 & -2 & 2 \\ 0 & 1 & -1 & 4 \end{array}\right) -1$$

$$\left(\begin{array}{ccc|c} 1 & -3 & 6 & -1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$x = 5$

$y = 6$

$z = 2$

2.18

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a)
$$\begin{vmatrix} x & 0 & x^2-2 \\ 0 & 1 & 1 \\ -1 & x & x-1 \end{vmatrix} = x \cdot (\cancel{x}-1-\cancel{x}) + (-1)\cdot(-(x^2-2))$$

$$= -x + x^2 - 2 = \underline{x^2 - x - 2}$$

$|A| = 0$: $x^2 - x - 2 = 0$

$$x = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2}$$

$x = \underline{2}$, $x = \underline{-1}$

$x \neq 2, -1$: $|A| \neq 0 \Rightarrow \text{rk } A = 3$

$x = 2, -1$: $|A| = 0 \Rightarrow \text{rk } A < 3$

$\underline{x=2}$: $A = \begin{pmatrix} \boxed{\begin{matrix} 2 & 0 \\ 0 & 1 \end{matrix}} & \begin{matrix} \cdot \\ \cdot \end{matrix} \\ \cdots & \cdot \end{pmatrix}$ $M_{12,12} = 2 \neq 0 \Downarrow \text{rk } A = 2$

$\underline{x=-1}$: $A = \begin{pmatrix} \boxed{\begin{matrix} -1 & 0 \\ 0 & 1 \end{matrix}} & \begin{matrix} \cdot \\ \cdot \end{matrix} \\ \cdots & \cdot \end{pmatrix}$ $M_{12,12} = -1 \neq 0 \Downarrow \text{rk } A = 2$

$$\text{rk}\begin{pmatrix} x & 0 & x^2-2 \\ 0 & 1 & 1 \\ -1 & x & x-1 \end{pmatrix} = \begin{cases} 3, & x \neq 2, -1 \\ 2, & x = 2, -1 \end{cases}$$

b)

$$\begin{vmatrix} t+3 & 5 & 6 \\ -1 & t-3 & -6 \\ 1 & 1 & t+4 \end{vmatrix} = (t+3)\cdot\big((t-3)(t+4)+6\big) + 1\cdot\big(5(t+4)-6\big) + 1\cdot\big(-30-6(t-3)\big)$$

$$= (t+3)\cdot\big((t-3)(t+4)\big) + 6\cdot(t+3) + \underbrace{5(t+4)-6-30-6(t-3)}_{-t\,\cancel{-18}\,+2}$$

$$= (t+3)(t-3)(t+4) + \overbrace{(5t+20)}^{5(t+4)}$$

$|A|=0$: $$= (t+4)\cdot\big[(t+3)(t-3)+5\big] = 0$$

$t=-4$ $\qquad t^2-9+5=0$

$t^2-4=0$

$t=\pm 2$

$t \neq 2,-2,-4$: $\operatorname{rk} A = 3$

$t=2$: $M_{12,13} = \begin{vmatrix} 5 & 6 \\ -1 & -6 \end{vmatrix} = -30+6 \neq 0$

$t=-2$: $M_{13,12} = \begin{vmatrix} 1 & 5 \\ 1 & 1 \end{vmatrix} = 1-5 \neq 0$

$t=-4$: $M_{12,12} = \begin{vmatrix} -1 & 5 \\ -1 & -7 \end{vmatrix} = 7+5 \neq 0$

$\operatorname{rk} A = 2$

$t=2$: $A = \begin{pmatrix} 5 & 5 & 6 \\ -1 & -1 & -6 \\ 1 & 1 & 6 \end{pmatrix}$ $\quad t=-2$: $\begin{pmatrix} 1 & 5 & 6 \\ -1 & -5 & -6 \\ 1 & 1 & 2 \end{pmatrix}$ $\quad t=-4$: $\begin{pmatrix} -1 & 5 & \cdot \\ -1 & -7 & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$

2.20

a) $-2x_1 - 3x_2 + x_3 = 3$

$4x_1 + 6x_2 - 2x_3 = 1$

$$\hat{A} = \left(\begin{array}{ccc|c} -2 & -3 & 1 & 3 \\ 4 & 6 & -2 & 1 \end{array}\right)$$

A

$$\left.\begin{array}{l} M_{12,12} = -12 + 12 = 0 \\ M_{12,13} = 4 - 4 = 0 \\ M_{12,23} = 6 - 6 = 0 \end{array}\right\} \quad rk\, A = \underline{1}$$

all 2-minors $= 0$
there is a 1-minor $\neq 0$

$M_{12,14} = -2 - 12 = -14 \neq 0 \Rightarrow rk\, \hat{A} = \underline{2}$

no solutions

c) ~~$x_1 = 2x_2$~~

x_3 free

$$\hat{A} = \left(\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 1 \\ 2 & 1 & -1 & 3 & 3 \\ 1 & 5 & -8 & 1 & 1 \\ 4 & 5 & -7 & 7 & 7 \end{array}\right)$$

$M_{1234,1245} = 0$

$rk\, \hat{A} = rk\, A = 3$

$$|A| = \begin{vmatrix} 1 & -1 & 2 & 1 \\ 2 & 1 & -1 & 3 \\ 1 & 5 & -8 & 1 \\ 4 & 5 & -7 & 7 \end{vmatrix} \cdot(-2) \cdot(-1) \cdot(-4) = \begin{vmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & -5 & 1 \\ 0 & 6 & -10 & 0 \\ 0 & 9 & -15 & 3 \end{vmatrix}$$

$$= 1 \cdot (-90 + 90) + 3 \cdot (-30 + 30) = \underline{0}$$

$M_{123,123} = 1 \cdot 0 = 0 \qquad M_{123,124} = 1 \cdot \begin{vmatrix} 3 & 1 \\ 6 & 0 \end{vmatrix} = -6 \neq 0$

$$M_{123,124} = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 3 \\ 1 & 5 & 1 \end{vmatrix} = 1\cdot(-14) - 2\cdot(-6) + 1(-4)$$

$$= -14 + 12 - 4 = \underline{-6} \neq 0$$

$\text{rk } A = \underline{3}$ $\qquad$ $\text{rk } \hat{A} = \underline{3}$

<u>Conclusion:</u> Consistent

$\#$ free vars $= n - \text{rk } A = 4 - 3 = \underline{1}$

<u>Solution:</u> $\left.\begin{array}{l}\text{Take eqn. (1), (2), (3)} \\ \text{Solve for vars. (1), (2), (4)}\end{array}\right\} M_{123,124}$

$$\begin{array}{cl} & \quad 1 \qquad\quad 2 \qquad\qquad\quad 4 \\ 1 & x_1 - x_2 \qquad + x_4 = 1 - 2x_3 \\ 2 & 2x_1 + x_2 \qquad + 3x_4 = 3 + x_3 \\ 3 & x_1 + 5x_2 \qquad + x_4 = 1 + 8x_3 \end{array}$$

$$\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 3 \\ 1 & 5 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 - 2x_3 \\ 3 + x_3 \\ 1 + 8x_3 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 3 \\ 1 & 5 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 - 2x_3 \\ 3 + x_3 \\ 1 + 8x_3 \end{pmatrix}$$

det. of this matrix is $M_{123,124} \neq 0$.

3.4

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i) $\underline{a}, \underline{b}, \underline{c}$ lin. independent $\Rightarrow$ $\underline{a}+\underline{b}, \underline{b}+\underline{c}, \underline{a}+\underline{c}$ are lin. independent

Proof:

Assume $\underline{a}, \underline{b}, \underline{c}$ lin. independent.

Check if $\underline{a}+\underline{b}, \underline{b}+\underline{c}, \underline{a}+\underline{c}$ are lin. independent.

$$x \cdot (\underline{a}+\underline{b}) + y \cdot (\underline{b}+\underline{c}) + z(\underline{a}+\underline{c}) = \underline{0}$$

$$(x+z) \cdot \underline{a} + (x+y) \cdot \underline{b} + (y+z) \cdot \underline{c} = \underline{0}$$

Since $\underline{a}, \underline{b}, \underline{c}$ are lin. independent,

$$x+z=0, \quad x+y=0, \quad y+z=0$$

$$z=-x, \quad y=-x \; : \quad y+z=(-x)+(-x)=0$$

$$-2x=0$$

$$z=0 \qquad y=0 \qquad x=0$$

Only the trivial Solution

$x=y=z=0$ $\Rightarrow$ $\underline{a}+\underline{b}, \underline{b}+\underline{c}, \underline{a}+\underline{c}$ are lin. independent

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3.14

$\underline{w} = x_1 \cdot \underline{v_1} + x_2 \underline{v_2} + x_3 \underline{v_3}$

$$\left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{array}\right) \begin{array}{c} 1 \\ 2 \end{array} \rightarrow \left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{array}\right) -3$$

$$\left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{array}\right)$$

$h = 5$: inf. many sol. (consistent) yes **3**

$h \neq 5$: no solutions (inconsistent) no

3.15

Is $\underline{v_1}, \underline{v_2}, \underline{v_3}$ lin. independent?

$$\begin{vmatrix} 2 & 1 & h+1 \\ 3 & 2 & h \\ -1 & 1 & h-2 \end{vmatrix} = 2\left(2(h-2)-h\right)$$
$$-3\left(\cancel{h}-2-(\cancel{h}+1)\right)$$
$$-1\left(h-2(h+1)\right)$$

$$= 4h - 8 - 2h + 9 + h + 2$$

$|A| = 0$: $= \underline{3h + 3} = 0 \quad h = -1$

$|A| = 0 \Leftrightarrow h = -1 \Rightarrow$ no

$|A| \neq 0 \Leftrightarrow h \neq -1 \Rightarrow$ yes **4**

3.11 A any matrix: $rk(A^TA) = rk(A)$

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a) $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \\ 4 & 8 \end{pmatrix}$ $\quad A^TA = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \\ 4 & 8 \end{pmatrix}$

$$= \begin{pmatrix} 30 & 60 \\ 60 & 120 \end{pmatrix}$$

$$|A^TA| = 30 \cdot 120 - 60^2 = 0$$
$$= 3600 - 3600$$
$$\underline{rk(A^TA) = 1 = rk(A)}$$

3.12 A $m \times n$-matrix

<u>Null space</u> of A = all $\underline{x}$ s.t. $A\underline{x} = \underline{0}$

i) Show that Nullspace (A) = Nullspace (A^TA)

<u>In other words:</u> $A\underline{x} = 0 \iff (A^TA)x = \underline{0}$

If $A\underline{x} = \underline{0}$, then $A^T(Ax) = A^T \cdot \underline{0}$
$(A^TA)\underline{x} = \underline{0}$

If $(A^TA)\underline{x} = 0$, then $\underline{x}^T(A^TA)\underline{x} = \underline{x}^T 0$
$(A\underline{x})^T \cdot (A\underline{x}) = \underline{0}$
Put $\underline{y} = A\underline{x}$: $\quad y^T \cdot y = \underline{0}$

$$(y_1\, y_2 \dots) \cdot \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = y_1^2 + y_2^2 + \dots + y_n^2 = 0$$

$A\underline{x} = \underline{0} \Leftarrow \underline{y} = \underline{0} \Leftarrow y_1 = y_2 = \dots = y_n = 0$

2) Use this to prove $rk(A) = rk(A^TA)$.

Know from 1) $A\underline{x} = \underline{0} \Leftrightarrow (A^TA)\underline{x} = \underline{0}$

$\underset{m \times n}{A}$

$\underset{n \times n}{A^TA} \leftarrow \underset{n \times m}{} \; \underset{m \times n}{}$

degrees of freedom = # degrees of freedom

$n - rk\,A = n - rk(A^TA)$

$\underline{rk\,A} = \underline{rk(A^TA)}$