LECTURE 9

EIVIND ERIKSEN, OCT 15 2015

GRA 6035 MATHEMATICS

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Plan:

1. Envelope theorems
2. Bordered Hessians

Reading:

[MEJ 19.2-19.3, (19.4-19.6)

Reminder: Plenary Session on Monday.

1. Envelope theorems

When we have an optimization problem that depends on parameters, how will the optimal solution change if the parameter change?

Unconstrained case:

Ex: max $f(x;a) = -x^2 + 2ax + 4$ ← unconstrained problem with parameter a

For a given value of a, solve the optimization problem

$$\boxed{\begin{array}{ll} x^*(a) & : \text{maximizer} \\ f^*(a) = f(x^*(a)) & : \text{optimal value function} \end{array}}$$

Solution:

Stationary pts: $f'_x = -2x + 2a = 0$

$\underline{x = a}$

Is $x = a$ max?

$f = -x^2 + 2ax + 4$

$f''_{xx} = -2 \Rightarrow f$ is concave

Therefore $x^*(a) = a$ is maximizer

$$f^*(a) = -a^2 + 2a \cdot a + 4$$
$$= \underline{\underline{a^2 + 4}}$$

$$\frac{df^*(a)}{da} = 2a$$

$a = 1: \quad f^*(1) = 5$

$a = 2: \quad f^*(2) = 8$

$$\boxed{\text{Envelope thm:} \quad \frac{df^*(a)}{da} = \frac{\partial f}{\partial a}\left(x = x^*(a)\right)}$$

Ex: $f = -x^2 + 2ax + 4$

$$\frac{\partial f}{\partial a} = 2x \qquad \frac{\partial f}{\partial a}\left(x = x^*(a)\right) = \underline{2x^*(a)}$$

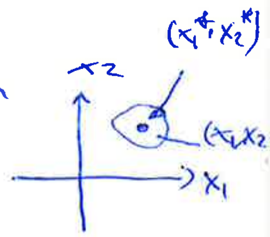
$$\frac{df^*(a)}{da} = 2x^*(a) = \underline{2a}$$

Interpretation:

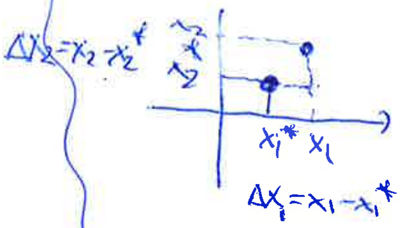
If we increase a with 1, $f^*(a)$ will increase with $\simeq 2a$.

Linear approximation of a function

Let $f(x_1, \dots, x_n)$ be a C^1 function in a neighbourhood at the point $(x_1^*, \dots, x_n^*)$. Then the linear approximation of f is



$$f(x_1, \dots, x_n) \simeq f(x_1^*, \dots, x_n^*) + \frac{\partial f}{\partial x_1}(x_1^*, \dots, x_n^*) \cdot (x_1 - x_1^*) + \frac{\partial f}{\partial x_2}(x_1^*, \dots, x_n^*) \cdot (x_2 - x_2^*) + \dots + \frac{\partial f}{\partial x_n} \cdot (x_1^*, \dots, x_n^*) \cdot (x_n - x_n^*)$$



Ex: $f(x,y) = e^{xy}$ at $(0,0)$ and $(1,1)$

$f(0,0) = \underline{1}$

$f'_x = e^{xy} \cdot y \qquad f'_x(0,0) = \underline{0}$

$f'_y = e^{xy} \cdot x \qquad f'_y(0,0) = \underline{0}$

When (x,y) close to $(0,0)$:

$$f(x,y) \simeq 1 + 0 \cdot (x-0) + 0 \cdot (y-0)$$
$$= 1$$

$f(1,1) = e \qquad f'_x(1,1) = e \qquad f'_y(1,1) = e$

When (x,y) close to $(1,1)$:

$$f(x,y) \simeq e + e \cdot (x-1) + e \cdot (y-1)$$
$$= ex + ey - e$$

Ex: $f(x,y) = x^2 + 3y^2 - 2x - 6y + 7$

$\underline{(x^*, y^*) = (1,1)}$: $\quad f(1,1) = 3$

(x,y) close to $(1,1)$:

$$f(x,y) \simeq f(1,1) + \frac{\partial f}{\partial x}(1,1)(x-1) + \frac{\partial f}{\partial y}(1,1)(y-1)$$

$$= 3 + 0 \cdot (x-1) + 0 \cdot (y-1)$$

$$= \underline{3} \leftarrow \text{linear approximation of } f \text{ at } (1,1).$$

$\frac{\partial f}{\partial x} = 2x - 2 \qquad \frac{\partial f}{\partial x}(1,1) = 0$

$\frac{\partial f}{\partial y} = 6y - 6 \qquad \frac{\partial f}{\partial y}(1,1) = 0$

$\underline{(x^*, y^*) = (0,0)}$: $\quad f(0,0) = 7$

$\frac{\partial f}{\partial x}(0,0) = -2$

$\frac{\partial f}{\partial y}(0,0) = -6$

$$f(x,y) \simeq f(0,0) + \frac{\partial f}{\partial x}(0,0) \cdot (x-0) + \frac{\partial f}{\partial y}(0,0) \cdot (y-0)$$

$$= 7 + (-2) \cdot (x-0) + (-6) \cdot (y-0)$$

$$= \underline{7 - 2x - 6y} \leftarrow$$

Linear approximation of f close to $(0,0)$.

Envelope theorem (unconstrained case):

If max/min $f(x_1,\dots,x_n;\,a)$ has maximizer / minimizer $\underline{x}^*(a)$, then

$$\underline{x}^*(a) = (x_1^*(a), x_2^*(a), \dots, x_n^*(a))$$

$$\frac{df^*(a)}{da} = \frac{\partial f}{\partial a}\left(\underline{x} = \underline{x}^*(a)\right)$$

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Ex: $\max\ \pi(x,y;p,q) = px + qy - (500 + 4x + 2y + 0.04x^2 - 0.01xy + 0.01y^2)$

Find linear approximation of $\underline{\pi^*(p,q)}$ around $\underline{(p,q) = (10,13)}$

Envelope thm:

$$\frac{\partial \pi^*(p,q)}{\partial p} = \frac{\partial \pi}{\partial p}\left(\underline{x}^*(p,q)\right) = \underline{x^*(p,q)} = \frac{40}{3}p + \frac{20}{3}q - \frac{200}{3}$$

$$\frac{\partial \pi^*(p,q)}{\partial q} = \frac{\partial \pi}{\partial q}\left(\underline{x}^*(p,q)\right) = \underline{y^*(p,q)}$$

$$= \frac{20}{3}p + \frac{160}{3}q - \frac{400}{3}$$

Stationary pts:

$$\pi'_x = p - 4 - 0{,}08x + 0{,}01y = 0$$

$$\pi'_y = q - 2 + 0{,}01x - 0{,}02y = 0$$

$$\left.\begin{aligned} 8x - y &= 100p - 400 \\ -x + 2y &= 100q - 200 \end{aligned}\right\}$$ One unique solution $x = x^*(p,q) = \dots$, $y = y^*(p,q) = \dots$

$$H(\pi) = \begin{pmatrix} -0.08 & 0.01 \\ 0.01 & -0.02 \end{pmatrix} \quad \left.\begin{aligned} D_1 &= -0.08 < 0 \\ D_2 &= 0.0016 - 0.0001 > 0 \end{aligned}\right\} \text{concave}$$

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Constrained case:

If the Lagrange problem

$$\max/\min \ f(x_1,\dots,x_n;a) \quad \text{when} \quad \begin{cases} g_1(\underline{x};a)=0 \\ \vdots \\ g_m(\underline{x};a)=0 \end{cases}$$

has an ordinary maximizer/minimizer

$$(x_1^*(a), x_2^*(a), \dots, x_n^*(a); \lambda_1^*(a), \lambda_2^*(a), \dots, \lambda_m^*(a))$$

then

$$\frac{df^*(a)}{da} = \frac{\partial \mathcal{L}}{\partial a}(x_1^*(a), x_2^*(a), \dots, x_n^*(a); \lambda_1^*(a), \dots, \lambda_m^*(a))$$

Similar in the Kuhn-Tucker case.

Ex: $\boxed{\max \ x+3y \quad \text{when} \quad x^2+y^2 \le 10}$

i) Solve the problem.

ii) Find an approximation of $\max \ x+4y$ when $x^2+y^2 \le 10$.

We consider this a problem with a parameter:

$$\max \ x+ay \quad \text{when} \quad x^2+y^2-10 \le 0$$

$$\mathcal{L} = x+ay-\lambda\cdot(x^2+y^2-10)$$

$\mathcal{L}'_x = 1-\lambda\cdot 2x = 0$ $\quad x^2+y^2-10 \le 0 \quad \lambda \ge 0$ and $\lambda\cdot(x^2+y^2-10)=0$

$\mathcal{L}'_y = a-\lambda\cdot 2y = 0$

$x = \frac{1}{2\lambda}$ $\quad x^2+y^2<10 \Rightarrow \lambda=0$ not possible

$y = \frac{a}{2\lambda}$ $\quad x^2+y^2=10$ $\quad 2\lambda = +\frac{\sqrt{1+a^2}}{\sqrt{10}}$

$$\frac{1}{(2\lambda)^2}+\frac{a^2}{(2\lambda)^2}=10 \qquad \frac{1+a^2}{4\lambda^2}=10 \Rightarrow 4\lambda^2 = \frac{1+a^2}{10}$$

$$x = \frac{1}{2\lambda} = \sqrt{\frac{10}{1+a^2}} = x^*(a)$$

$$y = \frac{a}{2\lambda} = a\cdot\sqrt{\frac{10}{1+a^2}} = \sqrt{\frac{10a^2}{1+a^2}} = y^*(a)$$

$$\lambda = \frac{1}{2}\sqrt{\frac{1+a^2}{10}} = \lambda^*(a)$$

ordinary pt, cand. for max

$x^2+y^2 \le 10$ bounded set

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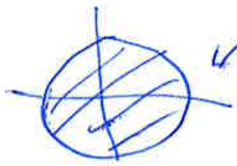
there is a a. max by EVT.



NDCQ is satisfied for all admissable pt's

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the point $(x,y;\lambda)$ is the maximizer

a=3:

$x^*(3) = 1$

$y^*(3) = 3$

$\lambda^*(3) = 1/2$

$f^*(3) = 1 + 3\cdot 3 = 10$

a=4:

Envelopeth:

$$\frac{df^*(a)}{da} = \frac{\partial L}{\partial a}\left(x^*(a);\lambda^*(a)\right) = y^*(a) = \sqrt{\frac{10a^2}{1+a^2}}$$

$f^*(3)=10$ $\quad \dfrac{df^*(a)}{d(a)} = \sqrt{\dfrac{10a^2}{1+a^2}}$

$$\left.\frac{df^*(a)}{da}\right|_{a=3} = 3$$

Linear approximation of $f^*(a)$ at $a=3$:

$$f^*(a) \simeq f^*(3) + \left.\frac{\partial f^*(a)}{\partial a}\right|_{a=3}\cdot(a-3)$$

$$= 10 + 3\cdot(a-3) = 10+3a-9 = 1+3a$$

$$f^*(4) \simeq 10 + 3\cdot(4-3) = 13$$

Exact solution:

$x^*(4) = \sqrt{\frac{10}{17}}$

$y^*(4) = \sqrt{\frac{160}{17}}$

$f^*(4) = x^*(4) + 4\cdot y^*(4)$

$= \sqrt{\frac{10}{17}} + 4\cdot\sqrt{\frac{160}{17}}$

$\simeq 13{,}04$

(2) Bordered Hessians

Lagrange problem: max/min $f(\underline{x})$ when $\begin{cases} g_1(\underline{x}) = a_1 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases}$

$$\mathcal{L} = f(\underline{x}) - \lambda_1 \cdot (g_1(\underline{x})) - \lambda_2 \cdot g_2(\underline{x}) - \dots - \lambda_m g_m(\underline{x})$$

$$\boxed{\begin{array}{ll} \mathcal{L}'_{x_1} = 0 & g_1(\underline{x}) = a_1 \\ \mathcal{L}'_{x_2} = 0 & \vdots \\ \vdots & g_m(\underline{x}) = a_m \\ \mathcal{L}'_{x_n} = 0 & \end{array}}$$

Assume $(x_1^*, x_2^*, \dots, x_n^*; \lambda_1^*, \lambda_2^*, \dots, \lambda_m^*)$. solution

is an ordinary candidate pt.

We can try to classify this point as local max/local min.

Bordered Hessian:

$$B = \left(\begin{array}{c|c} 0 & J \\ \hline J^T & H(\mathcal{L})_{\underline{x}} \end{array}\right)$$

$(m+n) \times (m+n)$ - matrix

$$J = \begin{pmatrix} \partial g_1/\partial x_1 & \partial g_1/\partial x_2 & \dots \\ \vdots & & \\ \partial g_m/\partial x_1 & \partial g_m/\partial x_2 & \dots \end{pmatrix}$$

$m \times n$ - matrix

$$H(\mathcal{L})_{\underline{x}} = \begin{pmatrix} \mathcal{L}''_{x_1x_1} & \mathcal{L}''_{x_1x_2} & \dots & \mathcal{L}''_{x_1x_n} \\ \vdots & & & \\ \mathcal{L}''_{x_1x_n} & \dots & \dots & \mathcal{L}''_{x_nx_n} \end{pmatrix}$$

$n \times n$ - matrix

In the Kuhn-Tucker case:

Similar, but we only include rows in J corresponding to constraints where the pt. $\underline{x}^*, \underline{\lambda}^*$ is binding (holds with equality), and m = # such rows.

Ex: max $x+3y$ when $x^2+y^2=10$

Cand. pt: $(x,y;\lambda) = (1,3;\,1/2)$

$$\mathcal{L} = x+3y-\lambda\cdot(x^2+y^2)$$

$$\text{FOC}\left\{\begin{array}{l} \mathcal{L}'_x = 1-\lambda\cdot 2x = 0 \\ \mathcal{L}'_y = 3-\lambda\cdot 2y = 0 \end{array}\right.$$

$$\text{C} \qquad x^2+y^2 = 10$$

Solve to find

$(1,3;\,1/2)$ $\quad f=10$

$(-1,-3;\,-1/2)$ $\quad f=-10$

Is this a local max?

$$\mathcal{L}''_{xx} = -2\lambda \qquad \mathcal{L}''_{xy} = 0 \qquad \mathcal{L}''_{yy} = -2\lambda$$

$$g'_x = 2x \qquad g'_y = 2y$$

$$B = \left(\begin{array}{c|cc} 0 & 2x & 2y \\ \hline 2x & -2\lambda & 0 \\ 2y & 0 & -2\lambda \end{array}\right)$$

$n=2$ (var's) $\quad$ $m=1$ (constraints) $\quad$ $\}\ n+m=3$

$$B(1,3;\,1/2) = \begin{pmatrix} 0 & 2 & 6 \\ 2 & -1 & 0 \\ 6 & 0 & -1 \end{pmatrix}$$

Condition in case $n-m=1$:

$|B|$ same sign as $(-1)^n$: local max

$\quad$ – " – $\quad$ $(-1)^m$: local min

In the example:

$n=2,\ m=1$: $\quad |B|>0 \longrightarrow$ local max

$\qquad\qquad\qquad |B|<0 \longrightarrow$ local min

$$|B| = \begin{vmatrix} 0 & 2 & 6 \\ 2 & -1 & 0 \\ 6 & 0 & -1 \end{vmatrix} = -2\cdot(-2-0)+6\cdot(0+6)$$

$$\shortparallel$$

$$|B(1,3;1/2)|$$

$$= 4+36 = \underline{40} > 0$$

$|B|$ positive $\Rightarrow$ $(1,3;1/2)$ is $\underline{\text{local max}}$

$\shortparallel$

Same sign as

$(-1)^n = (-1)^2 = +1$

ii) $\underline{\text{The case } n-m \text{ arbitrary:}}$

Compute the bordered Hessian $B(x_1^*,..,x_n^*;\lambda_1^*,..,\lambda_m^*) = B$, and compute the last $\underline{n-m}$ leading principal minors of this matrix.

$$\boxed{\begin{array}{ll} \text{Signs are alternating } \underline{\text{and}} \text{ the last sign is } (-1)^n & \rightarrow \underline{\text{local max}} \\ \text{Signs are all equal to the sign of } (-1)^m & \rightarrow \underline{\text{local min}} \end{array}}$$

$\underline{\text{Ex:}}$ max/min $\underline{x^2y^2z^2}$ when $x^2+y^2+z^2=3$

$n=3$ (var's) $\qquad m=1$ (constr.)

$n+m=4 \rightarrow B$ 4×4-matrix

$\underline{n-m=2}$

Compute D_3, D_4

$\underline{\text{Conditions:}}$

$$\boxed{\begin{array}{ll} D_3>0, D_4<0 & \rightarrow \text{local max} \\ D_3<0, D_4<0 & \rightarrow \text{local min.} \end{array}}$$

$$\mathcal{L} = x^2y^2z^2 - \lambda(x^2+y^2+z^2)$$

$$\left.\begin{aligned}
\mathcal{L}'_x &= 2xy^2z^2 - \lambda\cdot 2x = 0\\
\mathcal{L}'_y &= 2y\cdot x^2z^2 - \lambda\cdot 2y = 0\\
\mathcal{L}'_z &= 2zx^2y^2 - \lambda\cdot 2z = 0\\
x^2+y^2+z^2 &= 3
\end{aligned}\right\} \leftarrow \underline{\text{Solution}}$$

(i.e. candidate pt.)

$(x,y,z;\lambda) = (1,1,1;1)$

$\underline{f(1,1,1)=1}$

local max?

$$B = \begin{pmatrix} 0 & 2x & 2y & 2z \\ 2x & & & \\ 2y & & \boxed{H(\mathcal{L})_{\underline{x}}} & \\ 2z & & & \end{pmatrix} = \begin{pmatrix} 0 & 2 & 2 & 2 \\ 2 & 0 & 4 & 4 \\ 2 & 4 & 0 & 4 \\ 2 & 4 & 4 & 0 \end{pmatrix}$$

put in
$x=1$
$y=1$
$z=1$
$\lambda=1$

$$D_3 = -2\cdot(0-8)+2\cdot(8-0) = 16+16 = 32 > 0$$

$$D_4 = -2\cdot\begin{vmatrix} 2 & 2 & 2 \\ 0 & 4 & 4 \\ 4 & 0 & 4 \end{vmatrix} + 4\begin{vmatrix} 0 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 0 & 4 \end{vmatrix} - 4\cdot\begin{vmatrix} 0 & 2 & 2 \\ 2 & 0 & 4 \\ 2 & 4 & 4 \end{vmatrix}$$

$$= -2\left(4\cdot 0 + 4\cdot 8\right) + 4\cdot(2\cdot 0 + 4\cdot(-4)) - 4(-2\cdot 0 + 2\cdot 8)$$

$$= -64 - 64 - 64 = -192 < 0$$

$D_3 > 0,\ D_4 < 0 \implies$ $\underline{\underline{\text{local max in } (1,1,1)}}$