LECTURE 8

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GRA 6035 MATHEMATICS

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Plan:

1. Review: FOC's in Lagrange and Kuhn-Tucker problems
2. Lagrange multipliers
3. Second order conditions (SOC)

Reading:

[ME] 19.1, 19.4, (18.1 - 18.7)

1. Review:

First order conditions (FOC's) give candidates for max/min in Lagrange / Kuhn-Tucker problems:

i) Write the problem in std. form:

Lagrange case:

max/min $f(\underline{x})$ when

$$C\begin{cases} g_1(\underline{x}) = a_1 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases}$$

K-T case:

max $f(\underline{x})$ when

$$C\begin{cases} g_1(\underline{x}) \leq a_1 \\ \vdots \\ g_m(\underline{x}) \leq a_m \end{cases}$$

ii) Write down and solve:

Lagrange conditions

FOC + C

$$\boxed{\begin{array}{ll} \mathcal{L}'_{x_1} = 0 & g_1(\underline{x}) = a_1 \\ \vdots & \vdots \\ \mathcal{L}'_{x_n} = 0 & g_m(\underline{x}) = a_m \end{array}}$$

Kuhn-Tucker conditions

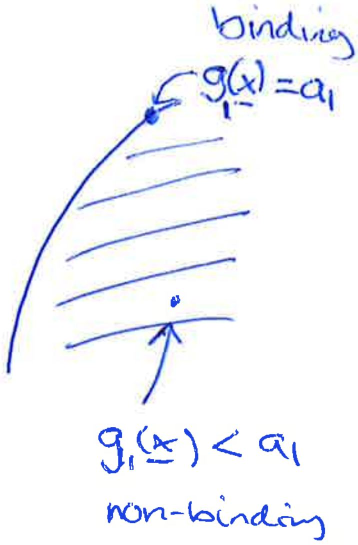
FOC + C + CSC

$$\boxed{\begin{array}{llll} \mathcal{L}'_{x_1} = 0 & g_1(\underline{x}) \leq a_1 & \lambda_1 \geq 0 & \lambda_1(g_1 - a_1) = 0 \\ \vdots & \vdots & \vdots & \vdots \\ \mathcal{L}'_{x_n} = 0 & g_m(\underline{x}) \leq a_m & \lambda_m \geq 0 & \lambda_m(g_m - a_m) = 0 \end{array}}$$

CSC: Complementary slackness conditions

KT problem, std. form

$$\max f(\underline{x}) \quad \text{s.t.} \quad \begin{cases} g_1(\underline{x}) \le a_1 \\ g_2(\underline{x}) \le a_2 \\ \vdots \\ g_m(\underline{x}) \le a_m \end{cases}$$



CSC for $g_1(\underline{x}) \le a_1$: $\lambda_1 \ge 0$ and $\underbrace{\lambda_1 \cdot (g_1(\underline{x}) - a_1) = 0}_{\lambda_1 = 0 \text{ or } g_1(\underline{x}) = a_1}$

In other words:

$g_1(\underline{x}) = a_1$ binding: $\lambda_1 \ge 0$

$g_1(\underline{x}) < a_1$ non-binding: $\lambda_1 = 0$

iii) Candidates for "max/min" optimum:

a) Solutions from ii) : Regular cond.
b) Admissable pts where NDCQ fails : special cond.

} list of candidate pts

C + $\boxed{\text{NDCQ fails}}$

Lagrange case: NDCQ

$$\text{rk}\left(\partial g_i/\partial x_j\right) = m$$

$$J = \begin{pmatrix} \partial g_1/\partial x_1 & \partial g_1/\partial x_2 & \cdots \\ \partial g_2/\partial x_1 & \partial g_2/\partial x_2 & \cdots \\ \vdots & & \end{pmatrix}$$

Kuhn-Tucker case: NDCQ

For an admissible pt $\underline{x}^*$

$$\text{rk}\left(\partial g_i/\partial x_j\right) = \#\text{rows}$$

include only rows for those constraints that are binding at $\underline{x}^*$, i.e. $g_i(\underline{x}^*) = a_i$

Main point:

Both in the Lagrange and Kuhn-tucker case, we have a systematic way to obtain a complete list of all possible candidates for max/min.

Two possibilities

- there is no max/min
- the max/min is contained in the list above

② Lagrange multipliers

Ex: max/min $f(x,y) = x + 3y$ when $x^2 + y^2 = 10$

Candidate pts:

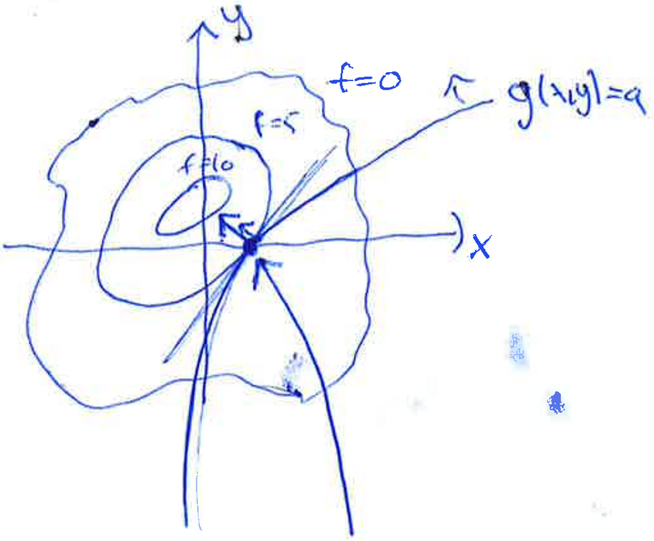
Regular: FOC + C — $(x, y; \lambda) = (1, 3; \tfrac{1}{2})$, $f = 10$
$(-1, -3; -\tfrac{1}{2})$, $f = -10$

Special: NDCQ fails + C — none

What is the interpretation of λ?

$\lambda > 0$: If you increase a, then f will increase
$\lambda < 0$: If you increase a, then f will decrease

$f(x,y)$ function of two variables



$f(x,y) = c$: All pts (x,y) s.t. $f(x,y) = c$

Gradient: $\nabla f = \begin{pmatrix} f'_x \\ f'_y \end{pmatrix}$

The gradient points in the direction where f increases most steeply.

$$\boxed{\max f(x,y) \text{ when } g(x,y) = a}$$

$$\nabla f = \lambda \nabla g$$

FOC: $f'_x = \lambda \cdot g'_x$
$f'_y = \lambda \cdot g'_y$

$$\Longleftrightarrow \begin{pmatrix} f'_x \\ f'_y \end{pmatrix} = \lambda \cdot \begin{pmatrix} g'_x \\ g'_y \end{pmatrix}$$

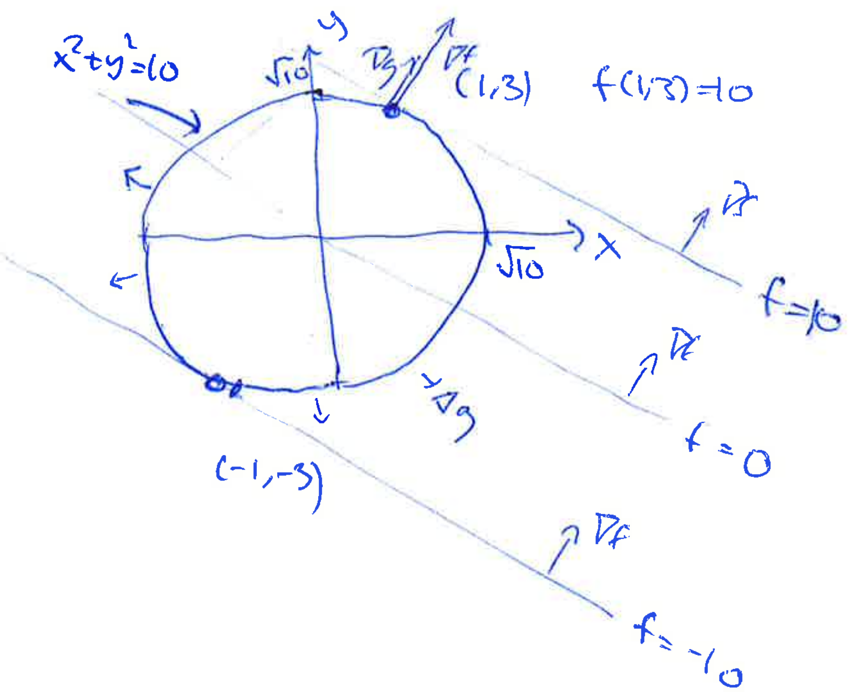
Interpretation:

Let us consider max/min $f(\underline{x})$ s.t. $\begin{cases} g_1(\underline{x}) = a_1 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases}$

Assume that $(\underline{x}^*; \underline{\lambda}^*)$ is a max that satisfy the Lagrange conditions FOC+C. Then the maximum value is $f(\underline{x}^*)$, and

$$\boxed{\lambda_i^* = \frac{\partial f(x^*)}{\partial a_i}}$$

Ex: max $f(x,y) = x + 3y$ when $x^2 + y^2 = 10$



Cond: $(1,3; \frac{1}{2})$ $\nabla f = \frac{1}{2} \nabla g$
$(-1,-3; -\frac{1}{2})$

Interpretation:

$x^2+y^2=10 \to x^2+y^2 = 10+\varepsilon$

$f(x^*,y^*) = 10 \to 10 + \frac{1}{2}\cdot\varepsilon$

$$\frac{\Delta f^*}{\Delta a} = \frac{\frac{1}{2}\varepsilon}{\varepsilon} = \frac{1}{2}$$

Similar picture when we have more constraints and for Kuhn-Tucker problems.

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Ex: $\min \underbrace{2x^2+y^2+3z^2}_{f(x,y,z)}$ when $\begin{array}{l} x-y+2z \geq 3 \\ x+y \geq 3 \end{array}$

Std. form $\boxed{\max \; -f(x,y,z) = -2x^2-y^2-3z^2 \quad \text{when} \quad \begin{array}{l} -x+y-2z \leq -3 \\ -x-y \leq -3 \end{array}}$

$$\mathcal{L} = -2x^2-y^2-3z^2 - \lambda_1(-x+y-2z) - \lambda_2\cdot(-x-y)$$

KT Conditions

FOC:

$\mathcal{L}'_x = -4x + \lambda_1 + \lambda_2 = 0$

$\mathcal{L}'_y = -2y - \lambda_1 + \lambda_2 = 0$

$\mathcal{L}'_z = -6z + 2\lambda_1 = 0$

C:

$x-y+2z \geq 3$

$x+y \geq 3$

CSC:

$\lambda_1 \geq 0$ and $\lambda_1\cdot(x-y+2z-3) = 0$

$\lambda_2 \geq 0$ and $\lambda_2\cdot(x+y-3) = 0$

$x-y+2z=3$ $x+y=3$	$x-y+2z=3$ $x+y>3$	$x-y+2z>3$ $x+y=3$	$x-y+2z>3$ $x+y>3$
$\lambda_1 \geq 0$ $\lambda_2 \geq 0$	$\lambda_1 \geq 0$ $\lambda_2 = 0$	$\lambda_1 = 0$ $\lambda_2 \geq 0$	$\lambda_1 = 0$ $\lambda_2 = 0$
FOC	FOC	FOC	FOC
	$\lambda_1 = 4x,\ \lambda_1 = -2y,\ \lambda_1 = 3z$ $x = \frac{\lambda_1}{4},\ y = -\frac{\lambda_1}{2}$ $z = \frac{\lambda_1}{3}$ $\frac{\lambda_1}{4} + \frac{\lambda_1}{2} + \frac{2}{3}\lambda_1 = 3$ $17\lambda_1 = 36$	$z=0$ $\lambda_2 = 4x = 2y$ $\Rightarrow y = 2x$ $x+y=3$ $x+2x=3$ $x=1, y=2, z=0, \lambda_1=0, \lambda_2=4$ not admissible	$x=y=z=0$ not admissible

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$\lambda_1 = \frac{36}{17}$

$x = \frac{36}{68} \quad y = -\frac{36}{34}$

$x+y = \frac{36}{68} - \frac{72}{68} = \frac{-36}{68}$

not admissible

$$\left.\begin{aligned} x-y+2z&=3\\ x+y&=3\\ \lambda_1&\geq 0\\ \lambda_2&\geq 0\\ -4x+\lambda_1+\lambda_2&=0\\ -2y-\lambda_1+\lambda_2&=0\\ -6z+2\lambda_1&=0 \end{aligned}\right\}$$

$\frac{\lambda_1+\lambda_2}{4} - \frac{-\lambda_1+\lambda_2}{2} + 2\cdot\frac{\lambda_1}{3} = 3 \quad |\cdot 12$

$\frac{\lambda_1+\lambda_2}{4} + \frac{-\lambda_1+\lambda_2}{2} = 3 \quad |\cdot 12$

$x = \frac{\lambda_1+\lambda_2}{4}$

$y = \frac{-\lambda_1+\lambda_2}{2}$

$z = \frac{\lambda_1}{3}$

$3(\lambda_1+\lambda_2) - 6(-\lambda_1+\lambda_2) + 8\lambda_1 = 36$

$3(\lambda_1+\lambda_2) + 6(-\lambda_1+\lambda_2) = 36$

$17\lambda_1 - 3\lambda_2 = 36$

$-3\lambda_1 + 9\lambda_2 = 36$

$48\lambda_1 = 144$

$\lambda_1 = \frac{144}{48} = \frac{12}{4} = \underline{3}$

$-9 + 9\lambda_2 = 36$

$\lambda_2 = \frac{45}{9} = \underline{5}$

Adm. pt: $\underline{(2, 1, 1;\ 3, 5)}$ $\quad -f = -12$ $\quad (f = 12)$

What about NDCQ:

$$\overset{g_1 \searrow}{-x+y-2z \le -3}$$
$$\underset{g_2 \nearrow}{-x-y} \quad \le -3$$

$$J = \begin{pmatrix} -1 & 1 & -2 \\ -1 & -1 & 0 \end{pmatrix}$$

NDCQ:

a) $\left.\begin{array}{l} -x+y-2z = -3 \\ -x-y = -3 \end{array}\right\}$ $\mathrm{rk}\begin{pmatrix} -1 & \boxed{\begin{matrix} 1 & -2 \\ -1 & 0 \end{matrix}} \end{pmatrix} = 2$ ok

b) $\left.\begin{array}{l} -x+y-2z = -3 \\ -x-y < -3 \end{array}\right\}$ $\mathrm{rk}\begin{pmatrix} -1 & 1 & -2 \end{pmatrix} = 1$ ok

c) $\left.\begin{array}{l} -x+y-2z < -3 \\ -x-y = -3 \end{array}\right\}$ $\mathrm{rk}\begin{pmatrix} -1 & -1 & 0 \end{pmatrix} = 1$ ok

d) $\left.\begin{array}{l} -x+y-2z < -3 \\ -x-y < -3 \end{array}\right\}$ no condition ok

ok means NDCQ is satisfied

No points where NDCQ fails in any of the cases.

Conclusion:

$$\boxed{\begin{array}{l} \min\ f = 2x^2+y^2+3z^2 \\ = \max\ -f = -2x^2-y^2-3z^2 \end{array}}$$

One regular cand: $(1,1,2\,;\,3,5)$ $\boxed{\begin{array}{l} -f=-12 \\ f=12 \end{array}}$

no special cand.

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either $(1,1,2)$ is min for f / max for $-f$

or there is no min for f / max for $-f$.

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③ Second order condition.

Thm: (Second order condition) = SOC

If $(x_1^*, x_2^*, \ldots, x_n^*; \lambda_1^*, \ldots, \lambda_m^*)$ is a regular candidate pt (satisfies FOC+C in the Lagrange case or FOC+C+CSC in the Kuhn-Tucker case), then we consider

$$\mathcal{L}(x_1, \ldots, x_n; \lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*) \quad \text{as a function in } \underline{x}.$$

If $\mathcal{L}(\underline{x}; \lambda^*)$ is convex, then $\underline{x}^*$ is a minimum pt.
If $\mathcal{L}(\underline{x}, \lambda^*)$ is concave, then $\underline{x}^*$ is a maximum pt.

Ex: max $-2x^2 - y^2 - 3z^2$ when $\begin{cases} -x + y - 2z \leq -3 \\ -x - y \leq -3 \end{cases}$

Candidate: $(x, y, z; \lambda_1, \lambda_2) = (2, 1, 1; \underline{3}, \underline{5})$

$$\mathcal{L} = -2x^2 - y^2 - 3z^2 - \lambda_1(-x + y - 2z) - \lambda_2(-x - y)$$
$$= -2x^2 - y^2 - 3z^2 - 3(-x + y - 2z) - 5(-x - y)$$
$$= -2x^2 - y^2 - 3z^2 + 8x + 2y + 6z$$

$$H(\mathcal{L}) = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -6 \end{pmatrix} \qquad \begin{aligned} D_1 &= -4 \\ D_2 &= 8 \\ D_3 &= -48 \end{aligned}$$

negative defn.
$\Downarrow$
concave
$\Downarrow$
$\underline{(2,1,1) \text{ is max}}$
(for $-f$)

What if SOC doesn't work:

① Is the admissible set bounded?

If the admissible set is bounded, then the problem has a solution by the extreme value theorem:

All continuous functions have a max and a min on any compact set (compact = closed and bounded)

Conclusion: One of the candidate pts is max/min.

② If not, try something else.

can $f \to \infty$ when $\underline{x}$ is adm?

" $f \to -\infty$ — " —

Ex: max/min $2x^2+y^2+3z^2$ when $\begin{cases} x-y+2z=3 \\ x+y=3 \end{cases}$

$f = 2x^2+y^2+3z^2 \ge 0$

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 1 & 1 & 0 & 3 \end{array}\right) \begin{matrix} \\ -1 \end{matrix}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 2 & -2 & 0 \end{array}\right)$$

z free

$x - y + 2z = 3$

$\underline{x = 3 - z}$

$2y - 2z = 0$

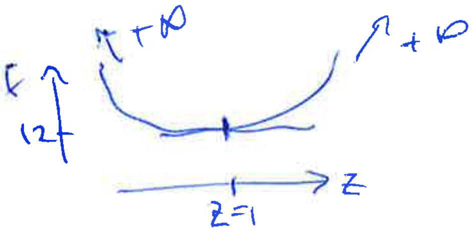
$\underline{y = z}$

$$\boxed{\begin{aligned} x &= 3-z \\ y &= z \\ z &= z \end{aligned}}$$

a line in 3-dim space $\mathbb{R}^3$

<u>not bounded</u>

$$\begin{aligned} f &= 2x^2+y^2+3z^2 \\ &= 2\cdot(3-z)^2 + z^2 + 3z^2 \\ &= 2(9-6z+z^2) + 4z^2 \\ &= 18 - 12z + 6z^2 \\ &= 6z^2 - 12z + 18 = 6(z-1)^2 + 12 \end{aligned}$$



$z=1$: $(x,y,z) = \underline{(2,1,1)}$ is minimum

<u>no max.</u>