LECTURE 7

GRA 6035 MATHEMATICS

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Plan:

1. Review: Unconstrained optimization
2. Constrained optimization and admissible pts.
3. Lagrange problems
4. Kuhn-Tucker problems

Reading:

[NE] 18.1–18.7, (12.3–12.5), 21.1

1. Unconstrained problem: max/min $f(\underline{x})$

Method:

1. Find stationary pts of f using FOC:

$$f'_{x_1} = f'_{x_2} = \dots = f'_{x_n} = 0$$

2. Classify each stationary pt $\underline{x}^*$ from 1 as local max/min or saddle pt using $H(f)(\underline{x}^*)$

   pos.def: local min
   neg.def: local max
   indef: saddle

3. Try to say something about global max/min using either

   i) f is convex/concave
   ii) AD HOC, by considering candidate pts, their values/types, and other considerations (can be difficult)

$f(x,y) = xy^2 + x^3y - xy$

Ex I: 6.1 iii) – see Plenary Session 2

Found stationary pts:
Classified them:

$$\begin{cases} (0,0),\ (0,1),\ (\pm 1,0) & \text{saddle pts} \\ \left(\sqrt{1/5},\ 2/5\right) & \text{local min} \\ \left(-\sqrt{1/5},\ 2/5\right) & \text{local max} \end{cases}$$

What about global max/min?

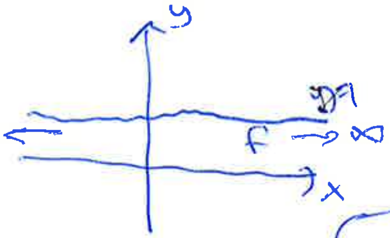
f not convex
f not concave

$f\left(-\frac{4}{25} - \frac{2}{25} + \frac{10}{25}\right)\frac{1}{\sqrt{5}} = \frac{4}{25}\cdot\frac{1}{\sqrt{5}}$

Global max: $\left(-\sqrt{1/5}, 2/5\right)$ is global max $f = \frac{4}{25}\cdot\frac{1}{\sqrt{5}}$

or

there is no global max



$f \to -\infty$

let us try $y=1$: $f(x,1) = x^3$

$\lim_{x\to\infty} f(x,1) = \infty$

$\lim_{x\to-\infty} f(x,1) = -\infty$

Conclusion: no max
no min

Ex 2: $f(x,y) = x^2y^3 + y^2 - 2y$

$f'_x = 2xy^3 = 0$

$f'_y = 3x^2y^2 + 2y - 2 = 0$

$x = 0$ or $y = 0$

$x=0$	$y=0$
$2y-2=0$	$-2=0$
$y=1$	$\Downarrow$
$\Downarrow$	no solution
$(0,1)$	

Stationary pts: $(0,1)$

$$H(f) = \begin{pmatrix} 2y^3 & 6xy^2 \\ 6xy^2 & 6x^2y+2 \end{pmatrix}$$

$$H(f)(0,1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad D_1 = 2 \quad D_2 = 4$$

pos. defn. $\Downarrow$ $(0,1)$ local min

Global max/min: no global max. (no local max)

Global min: $(0,1)$ $f(0,1) = -1$ no global min

convex? $D_1 = 2y^3$ can be both pos. and neg. $\Rightarrow$ not convex

Try: $x=+1$ $f = y^3 + y^2 - 2y \to -\infty$ when $y \to -\infty$

Try: $y=-1$ $f = -x^2 + 1 + 2 = 3 - x^2 \to -\infty$ as $x \to \pm\infty$.

(2) Constrained optimization

max/min $f(\underline{x})$ when $\underline{x}$ satisfy certain constraints

$\underbrace{}$

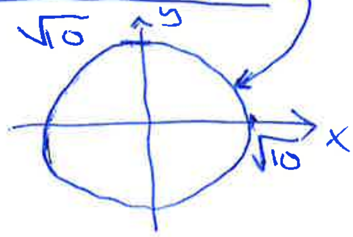
admissible pts = pts that satisfy the constraints

Ex:

max/min $x + 3y$ when $\underbrace{x^2+y^2=10}_{\text{equality constraint}}$ (circle center $(0,0)$ radius $\sqrt{10}$)

$x+3y$ = $f(x,y)$ objective function

admissible pts:



a) Equality constraints: Lagrange problem

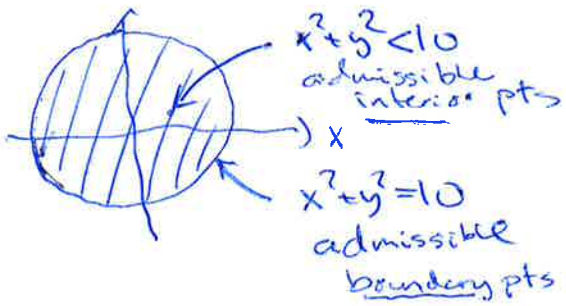
b) Constraints given by $\leq, \geq$: Kuhn-Tucker problem

(closed inequality constraints)

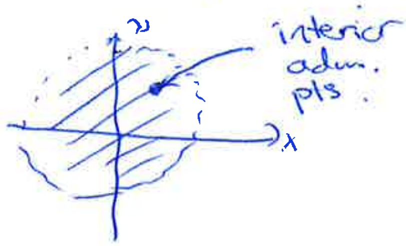
c) Open inequality constraints: Solved like unconstrained problems

(given by $<, >$)

Ex: max/min $f = x + 3y$ when $x^2+y^2 \leq 10$



Ex: max/min $f = x+3y$ when $x^2+y^2<10$



Admissible set:

D = set of admissible pts.

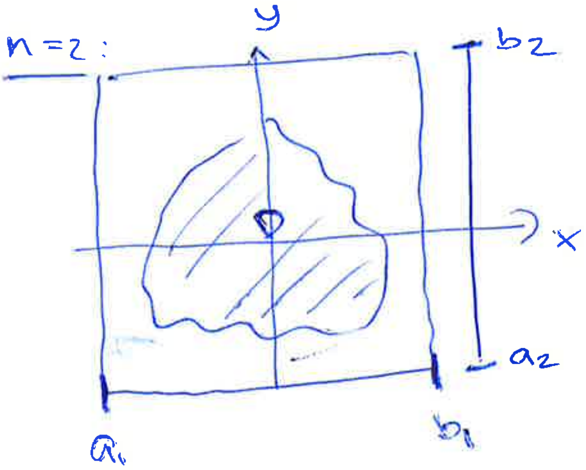
D is closed if it contains all its boundary pts. For Lagrange / KT-problems, D is closed.

D is bounded if there are number $a_1, \dots, a_n$, $b_1, \dots, b_n$ s.t.

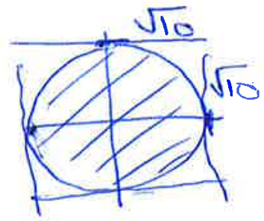
$$\begin{aligned} a_1 &\le x_1 \le b_1 \\ a_2 &\le x_2 \le b_2 \\ &\vdots \\ a_n &\le x_n \le b_n \end{aligned}$$

for all $(x_1, \dots, x_n)$ in D.

$n=2$:



Ex: $x^2+y^2 \le 10$

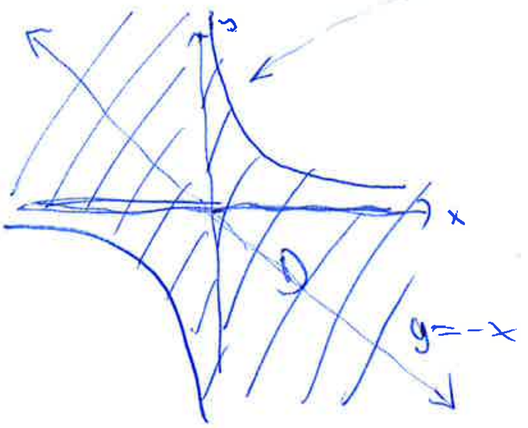


$D = \{(x,y) : x^2+y^2 \le 10\}$

$$\left.\begin{aligned} -\sqrt{10} &\le x \le \sqrt{10} \\ -\sqrt{10} &\le y \le \sqrt{10} \end{aligned}\right\} \text{ bounded}$$

$y=0$, x = anything
not bounded

Ex: $\boxed{xy \le 1}$



not bounded

$xy=1$
$y = 1/x$

$xy<1$

$x>0$: $y < 1/x$

$x<0$: $y > 1/x$

Extreme Value theorem

If f is a continuous function and D is a compact set (compact = closed and bounded), then max/min $f(\underline{x})$ s.t. $\underline{x}$ is in D has a solution (both a max and a min).

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③ Lagrange problems

Ex: max/min $\underbrace{x+3y}_{= f(x,y)}$ when $\underbrace{x^2+y^2}_{= g(x,y)} = \underbrace{10}_{= a}$

FOC: (first order condition)

$$\mathcal{L} = f(x,y) - \lambda \cdot g(x,y)$$
$$= x+3y - \lambda \cdot (x^2+y^2)$$

Lagrangian

FOC: $\mathcal{L}'_x = 1 - \lambda \cdot 2x = 0$

$\mathcal{L}'_y = 3 - \lambda \cdot 2y = 0$

C: $x^2+y^2 = 10$

FOC + C = Lagrange conditions

Solve FOC + C $\Rightarrow$ list of candidates for max/min.

$1 - 2\lambda \cdot x = 0 \Rightarrow x = \frac{1}{2\lambda}$, $\lambda \neq 0$

$3 - 2\lambda \cdot y = 0 \Rightarrow y = \frac{3}{2\lambda}$

$x^2+y^2 = 10 \Rightarrow \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{3}{2\lambda}\right)^2 = 10$

$\frac{10}{4\lambda^2} = \frac{1+9}{4\lambda^2} = 10$

$4\lambda^2 = 1 \quad \lambda^2 = 1/4 \quad \lambda = \pm 1/2$

$\lambda = 1/2$: $x = 1$, $y = 3$

$\lambda = -1/2$: $x = -1$, $y = -3$

Two candidate pts:

$(1, 3; 1/2)$ $(-1, -3; -\frac{1}{2})$

General Lagrange problem:

$$\max/\min \; f(\underline{x}) \quad \text{when} \quad \begin{cases} g_1(\underline{x}) = a_1 \\ g_2(\underline{x}) = a_2 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases}$$

$$f(\underline{x}) = f(x_1, \ldots, x_n)$$

Lagrangian: $\mathcal{L}(\underline{x}; \underline{\lambda}) = f(\underline{x}) - \lambda_1 \cdot (g_1(\underline{x})) - \lambda_2 \cdot g_2(\underline{x}) - \ldots$

$\ldots - \lambda_m g_m(\underline{x})$

$\underline{x}$: $x_1, \ldots, x_n$ $\quad$ $\underline{\lambda}$: $\lambda_1, \ldots, \lambda_m$

$\lambda_1, \ldots, \lambda_m$: Lagrange multipliers

$$\boxed{\textbf{FOC:} \quad \begin{aligned} \mathcal{L}'_{x_1} &= 0 \\ \mathcal{L}'_{x_2} &= 0 \\ &\vdots \\ \mathcal{L}'_{x_n} &= 0 \end{aligned}} \qquad \boxed{\textbf{C:} \quad \begin{aligned} g_1(\underline{x}) &= a_1 \\ g_2(\underline{x}) &= a_2 \\ &\vdots \\ g_m(\underline{x}) &= a_m \end{aligned}}$$

$n+m$ eqn's in $n+m$ var's

$$\underbrace{\text{FOC} + \text{C}}_{} = \text{Lagrange conditions}$$

Solutions of — " — gives candidates for max/min

Thm: Necessary conditions for Lagrange problems.

If a Lagrange problem has a solution $\underline{x}^*$, then either the NDCQ fails at $\underline{x}^*$ or there are $\underline{\lambda}^*$ s.t. $(\underline{x}^*; \underline{\lambda}^*)$ solves the Lagrange conditions.

Ex: max/min $x+3y$ when $x^2+y^2=10$

FOC+C $\longrightarrow$ Candidates

$(x,y;\lambda)=(1,3;\tfrac{1}{2})$ $\quad f=10$

$(x,y,\lambda)=(-1,-3;-\tfrac{1}{2})$ $\quad f=-10$

NDCQ: non degenerate constraint qualification

$$g(x,y)=\underline{x^2+y^2}=10$$

NDCQ: $\operatorname{rk}\begin{pmatrix}2x & 2y\end{pmatrix}=1$

$$\begin{pmatrix}2x & 2y\end{pmatrix} = \begin{pmatrix}g'_x & g'_y\end{pmatrix}$$

NDCQ fails: $\operatorname{rk}\begin{pmatrix}2x & 2y\end{pmatrix}<1$

$$\Updownarrow$$

$$\operatorname{rk}\begin{pmatrix}2x & 2y\end{pmatrix}=0$$

$$\Updownarrow$$

$$2x=2y=0$$

not admissible $x^2+y^2\neq 10$ $\longrightarrow$ $(x,y)=(0,0)$

$\boxed{\text{no admissible pts where NDCQ fails.}}$

Conclude:

$f=-10$

$(1,3)$, $(-1,-3)$ only candidates for max/min

$f=10$

EVT holds (D bounded)

$\Rightarrow$ $(1,3)$ is max, $(-1,-3)$ min

Ex: max y when $x^2+y^3=0$

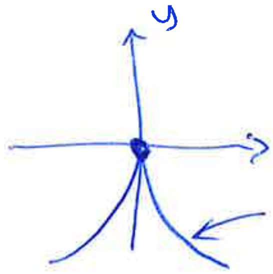
FOC+C: $\mathcal{L}=y-\lambda\cdot(x^2+y^3)$

$$\mathcal{L}'_x=-2\lambda x=0$$

$$\mathcal{L}'_y=1-3\lambda y^2=0$$

$$x^2+y^3=0$$

$x=0$	or	$\lambda=0$
$y=0$		$1=0$
$1=0$		no cand.
no solutions		



$x^2+y^3=0$

$y^3=-x^2$

$y=-\sqrt[3]{x^2}$

max y when (x,y) is admissible $=0$

Solution: $(0,0)$

NDCQ: $g=x^2+y^3=0$

$$\operatorname{rk}\begin{pmatrix}2x & 3y^2\end{pmatrix}=1$$

NDCQ fails: $\operatorname{rk}\begin{pmatrix}2x & 3y^2\end{pmatrix}=0$

$$2x=0,\ 3y^2=0$$

adm.! $x^2+y^3=0$ $\longrightarrow$ $(0,0)$

$$\boxed{\text{adm. pts where NDCQ fails} = (0,0)}$$

General NDCQ: $\begin{cases} g_1(\underline{x}) = a_1 \\ g_2(\underline{x}) = a_2 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases}$

NDCQ:
$$\operatorname{rk} \begin{pmatrix} \partial g_1/\partial x_1 & \partial g_1/\partial x_2 & \cdots & \partial g_1/\partial x_n \\ \partial g_2/\partial x_1 & \partial g_2/\partial x_2 & \cdots & \partial g_2/\partial x_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial g_m/\partial x_1 & \partial g_m/\partial x_2 & \cdots & \partial g_m/\partial x_n \end{pmatrix} = m$$

$\boxed{\text{If there are admissible pts s.t. NDCQ fails, then these are candidates for max/min.}}$

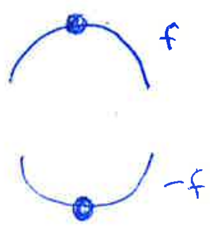
(4) Kuhn-Tucker problems

Standard form: $\max f(\underline{x})$ when $\begin{cases} g_1(\underline{x}) \le a_1 \\ g_2(\underline{x}) \le a_2 \\ \vdots \\ g_m(\underline{x}) \le a_m \end{cases}$

$f(\underline{x}) = f(x_1, \dots, x_n)$

Ex: $\min x+3y$ when $\underbrace{x^2+y^2 \le 10}_{\text{standard}}$

$= \max -(x+3y)$ when $x^2+y^2 \le 10$



Ex: $\underbrace{\max x+3y}_{\text{std.}}$ when $x^2+y^2 \ge 10 \quad | \cdot (-1)$

$-(x^2+y^2) \le -10$

$= \max x+3y$ when $-x^2-y^2 \le -10$

Consider $\max f(\underline{x})$ when $\begin{cases} g_1(\underline{x}) \le a_1 \\ \vdots \\ g_m(\underline{x}) \le a_m \end{cases}$

a Kuhn-Tucker problem in standard form.

$$\mathcal{L} = f(\underline{x}) - \lambda_1 \cdot g_1(\underline{x}) - \lambda_2 g_2(\underline{x}) - \dots - \lambda_m \cdot g_m(\underline{x})$$

$$\boxed{\underline{\text{FOC}}: \quad \begin{aligned} \mathcal{L}'_{x_1} &= 0 \\ \mathcal{L}'_{x_2} &= 0 \\ &\vdots \\ \mathcal{L}'_{x_n} &= 0 \end{aligned}} \qquad \boxed{C: \quad \begin{aligned} g_1(\underline{x}) &\le a_1 \\ &\vdots \\ g_m(\underline{x}) &\le a_m \end{aligned}} \qquad \boxed{\underline{\text{CSC}}: \ast}$$

Kuhn-Tucker conditions = FOC + C + CSC

CSC = $\underline{\text{complementary slackness conditions}}$

$$\boxed{\begin{array}{lll} \lambda_1 \ge 0 & \text{and} & \lambda_1 \cdot (g_1(\underline{x}) - a_1) = 0 \\ \lambda_2 \ge 0 & \text{"} & \lambda_2 \cdot (g_2(\underline{x}) - a_2) = 0 \\ \vdots & & \vdots \\ \lambda_m \ge 0 & \text{"} & \lambda_m \cdot (g_m(\underline{x}) - a_n) = 0 \end{array}}$$

$\lambda_1 \ge 0$ and $\lambda_1 \cdot (g_1(\underline{x}) - a_1) = 0$

$\lambda_1 \ge 0$ and $(\lambda_1 = 0 \text{ or } g_1(\underline{x}) = a_1)$

$\lambda_1 = 0$ or $(\lambda > 0, g_1(x) = a_1)$

Ex: $\max\ x+3y$ when $x^2+y^2 \le 10$

$$\mathcal{L} = x+3y - \lambda(x^2+y^2)$$

FOC: $\mathcal{L}'_x = 1 - \lambda \cdot 2x = 0$

$\mathcal{L}'_y = 3 - \lambda \cdot 2y = 0$

C: $x^2+y^2 \le 10 \iff x^2+y^2=10$ or $x^2+y^2<10$

CSC: $\lambda \ge 0,\ \lambda \cdot (x^2+y^2-10)=0 \iff \lambda \ge 0$ (for $x^2+y^2=10$), $\lambda = 0$ (for $x^2+y^2<10$)

KT conditions

C:	$x^2+y^2=10$	$x^2+y^2<10$
CSC:	$\lambda \ge 0$	$\lambda = 0$
FOC:	$1-\lambda \cdot 2x=0$ $3-\lambda\cdot 2y=0$	$1-\lambda\cdot 2x=0$ $3-\lambda\cdot 2y=0$
	Lagrange case:	**Unconstrained case:**
	$x=1, y=3; \lambda=1/2$	$\left.\begin{matrix}1=0\\3=0\end{matrix}\right\}$ no cand.

$(1,3;\ 1/2)$ only cand. which satisfy

$f=10$

KT cond = FOC + C + CSC

Thm: Necessary conditions for Kuhn-Tucker problems

If $\underline{x}^*$ solves a Kuhn-Tucker problem, then either $\underline{x}^*$ fails NDCQ or there are $\underline{\lambda}^*$ such that $(\underline{x}^*; \lambda^*)$ satisfy the Kuhn-Tucker conditions (= FOC + C + CSC).

NDCQ for KT-problems:

$$g_1(\underline{x}) \le a_1$$
$$\vdots$$
$$g_m(\underline{x}) \le a_m$$

Let $\underline{x}^*$ be an admissible pt. such that $g_i(\underline{x}) = a_i$ (the constraint is binding) for $i = i_1, i_2, \dots, i_k$, and $g_i(\underline{x}) < a_i$ for all other i (the constraint is not binding or non-binding). Then the NDCQ for $\underline{x}^*$ is

$$\operatorname{rk} \begin{pmatrix} \partial g_{i_1}/\partial x_1 & \frac{\partial g_{i_1}}{\partial x_2} & \cdots & \partial g_{i_1}/\partial x_n \\ \partial g_{i_2}/\partial x_1 & \partial g_{i_2}/\partial x_2 & \cdots & \partial g_{i_2}/\partial x_n \\ \vdots & \vdots & & \vdots \\ \partial g_{i_k}/\partial x_1 & \partial g_{i_k}/\partial x_2 & \cdots & \partial g_{i_k}/\partial x_n \end{pmatrix} = k$$

Include only the rows corresponding to constraints that are binding, ie holds with equality $g_i(\underline{x}) = a_i$