# LECTURE 6

# GRA 6035 MATHEMATICS

BI

EIVIND ERIKSEN, SEP 24, 2015

**Plan:**

1. Definiteness of quadratic forms (review)
2. Unconstrained optimization
3. Convex and concave functions

**Reading:**

[ME] 14.1–14.4, 14.8, 17.1–17.5

**Reminder:** Plenary Session 2 Mon Sep 28TH

Selected problems Lecture 4–6

## 1. Review: Definiteness

$Q(x_1, \dots, x_n)$ : quadr. form

$A$ : symm. matrix $(n \times n)$ s.t. $Q(\underline{x}) = \underline{x}^T A \underline{x}$

**Ex:** $Q = -x_1^2 + 4x_1x_2 + 3x_1x_3 - 5x_2^2 - 6x_3^2 - x_4^2 = \underline{x}^T \cdot A \cdot \underline{x}$, $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$

$$A = \begin{pmatrix} -1 & 2 & 3/2 & 0 \\ 2 & -5 & 0 & 0 \\ 3/2 & 0 & -6 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

**Leading principal minors:**

$D_1 = -1 < 0$

$D_2 = 1 > 0$

$D_3 = -6 \cdot D_2 + 3/2 \cdot 15/2 = \frac{45}{4} - \frac{24}{4} = \frac{21}{4} > 0$

$D_4 = ?$

Q is indefinite

# Criterion using principal minors

1) Compute leading principal minors $D_1, D_2, \dots, D_n$

2) Check:

$$D_1, D_2, \dots, D_n > 0 \iff A \text{ positive defn.}$$

$$\underbrace{D_1 < 0, D_2 > 0, D_3 < 0, \dots}_{\Updownarrow} \iff A \text{ negative definite}$$

$$(-1)^i D_i > 0 \text{ for all } i$$

3) If $D_1, \dots, D_n$ fails both of these patterns because of a wrong sign, then $A$ is indefinite

4) Otherwise, we need to compute all principal minors $\Delta_1, \Delta_2, \dots, \Delta_n$.

$$\Delta_1, \Delta_2, \dots, \Delta_n \geq 0 \iff A \text{ positive semidefinite}$$

$$\underbrace{\Delta_1 \leq 0, \Delta_2 \geq 0, \dots}_{\Updownarrow} \iff A \text{ negative semidefinite}$$

$$(-1)^i \Delta_i \geq 0 \text{ for all } i$$

---

Ex 1: $\underset{-1}{D_1} < 0, \quad \underset{1}{D_2} > 0, \quad \underset{21/4}{D_3} > 0, \quad D_4 = ?$

If $D_i < 0$ when $i$ is even, then $A$ is indefinite

If two $D_i$'s with $i$ odd have opposite signs, then $A$ is indefinite

## Why this criterion:

If $A$ is diagonal, it is $A = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$

Then

$$D_1 = \lambda_1$$
$$D_2 = \lambda_1 \cdot \lambda_2$$
$$D_3 = \lambda_1 \cdot \lambda_2 \cdot \lambda_3$$
$$\vdots$$
$$D_n = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \ldots \cdot \lambda_n$$

**Ex:** $Q = x^2 + 4xy - 8xz + 4y^2 - 16yz + z^2$

$$A = \begin{pmatrix} 1 & 2 & -4 \\ 2 & 4 & -8 \\ -4 & -8 & 1 \end{pmatrix}$$

$D_1 = \underline{1}$
$D_2 = \underline{0}$
$D_3 = 1 \cdot 0 + 8 \cdot 0 + (-4) \cdot 0 = \underline{0}$

Need to compute the remaining principal minors to find out if $A$ is $\left\{ \begin{array}{l} \text{pos. semidefin.} \\ \text{or indefinite} \end{array} \right.$

$\Delta_1^{11} = 1 \qquad \Delta_1^{22} = 4 \qquad \Delta_1^{33} = 1$

$\Delta_2^{12,12} = 0 \qquad \Delta_2^{13,13} = 1 - 16 = \underline{-15}$

$\Delta_2^{23,23} = \underline{-60}$

$$\Downarrow$$

$\Delta_3^{123,123} = 0$

$\underline{\underline{A \text{ is indefinite}}}$

## ② Unconstrained optimization

**Problem:** max / min $f(x_1, x_2, \dots, x_n) = f(\underline{x})$ when $\underline{x} = (x_1, \dots, x_n)$ is any point in $\mathbb{R}^n$ (n-dimensional space)

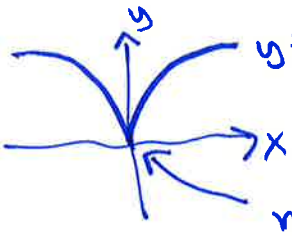
**We always assume $f$ is $C^2$:**

$f$ is $C^1$ if all partial derivatives $f'_{x_1}, f'_{x_2}, \dots, f'_{x_n}$ exist and are continuous

$f$ is $C^2$ if all second order partial derivatives $f''_{x_i x_j}$ exist and are continuous

**Ex:** $f(x) = \sqrt[3]{x^2} = x^{2/3}$

$f'(x) = (2/3) \cdot x^{2/3 - 1} = \frac{2}{3} x^{-1/3} = \dfrac{2}{3 \cdot \sqrt[3]{x}}, \quad x \neq 0$




$y = f(x) = \sqrt[3]{x^2}$
cusp.

no derivative at 0

**not $C^1$**

**Hard:** Find example that is $C^1$ but not $C^2$:

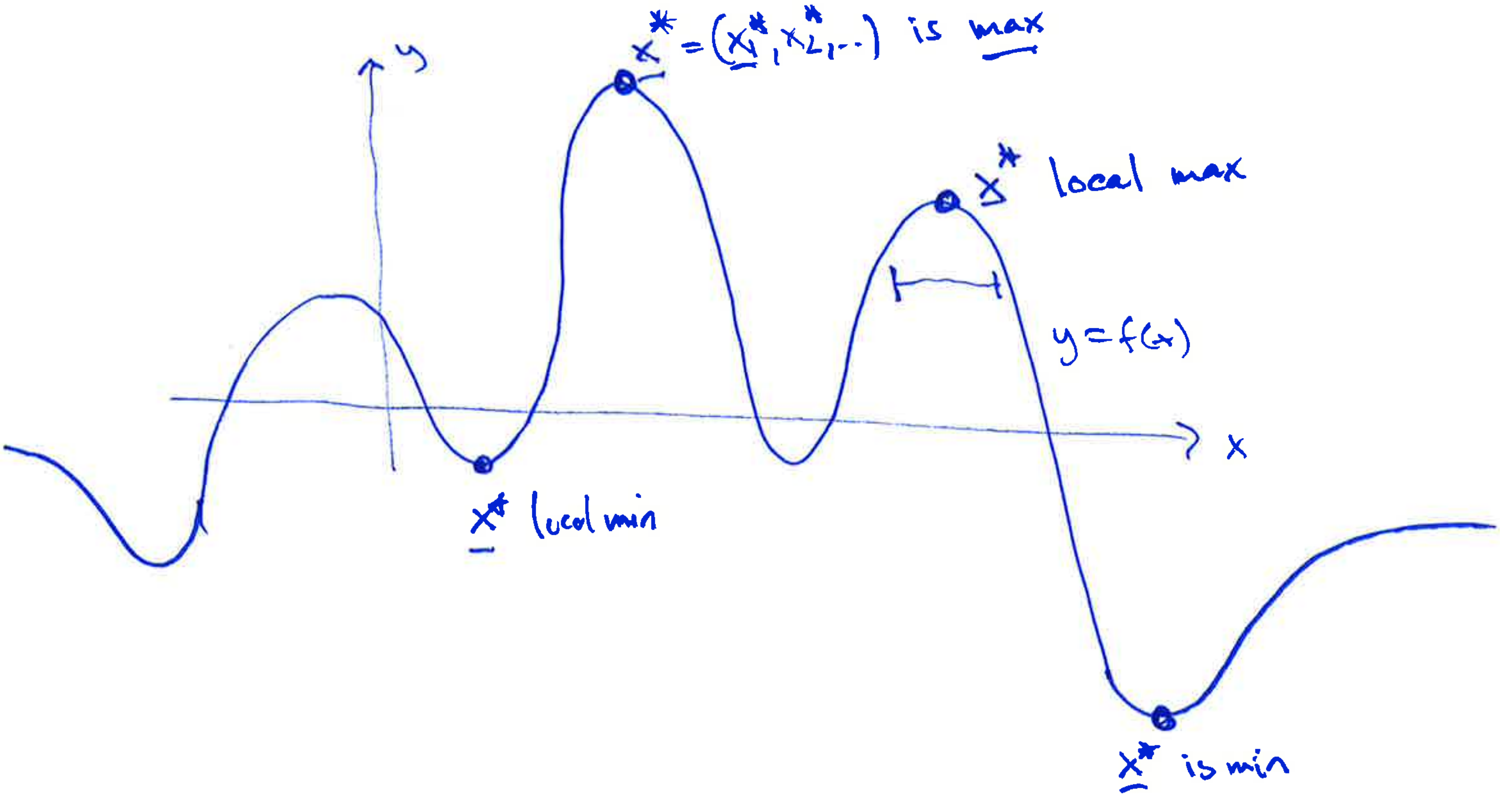
**Ex:** $f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

$C^1$ but $H(f)$ does not exist at $(0,0)$ (and cont.)

# Local / global max / min for $f(\underline{x})$

$\underline{x}^*$ **max** (global max) if $f(\underline{x}^*) \geq f(\underline{x})$ for all $\underline{x}$

$\underline{x}^*$ **min** (global min) if $f(\underline{x}^*) \leq f(\underline{x})$ — / —




$\underline{x}^*$ **local max** if $f(\underline{x}^*) \geq f(\underline{x})$ for all $\underline{x}$ close to $\underline{x}^*$

$\underline{x}^*$ **local min** if $f(\underline{x}^*) \leq f(\underline{x})$ — / — close to $\underline{x}^*$

## Derivatives:

**Ex:** $f(x,y) = x^3 + xy - y^3$

$f'_x = 3x^2 + y$

$f'_y = x - 3y^2$

first order partial derivatives

$f''_{xx} = 6x \qquad f''_{xy} = 1$

$f''_{yx} = 1 \qquad f''_{yy} = -6y$

second order partial derivatives

**Theorem:**

If $f$ is $C^2$, then

$$f''_{x_i x_j} = f''_{x_j x_i}$$

**Consequence:**

$f$ is $C^2 \Rightarrow H(f)$ is symmetric

**Hessian matrix of f:**

$$H(f) = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{pmatrix}$$

**In ex:**

$$H(f)(x,y) = \begin{pmatrix} 6x & 1 \\ 1 & -6y \end{pmatrix}$$

**Ex:** $f(x,y) = \ln(x^2+y^2+1)$

$$f'_x = \frac{1}{x^2+y^2+1} \cdot 2x = \frac{2x}{x^2+y^2+1}$$

$$f'_y = \frac{1}{x^2+y^2+1} \cdot 2y = \frac{2y}{x^2+y^2+1}$$

$$f''_{xx} = \frac{2 \cdot (x^2+y^2+1) - 2x \cdot 2x}{(x^2+y^2+1)^2} = \frac{-2x^2+2y^2+2}{(x^2+y^2+1)^2}$$

$$f''_{xy} = \frac{0 \cdot \cancel{x} - 2x \cdot 2y}{(x^2+y^2+1)^2} = \frac{-4xy}{(x^2+y^2+1)^2}$$

$$f''_{yy} = \frac{2x^2-2y^2+2}{(x^2+y^2+1)^2}$$

$$H(f) = \begin{pmatrix} \dfrac{-2x^2+2y^2+2}{(x^2+y^2+1)^2} & \dfrac{-4xy}{(x^2+y^2+1)^2} \\[2ex] \dfrac{-4xy}{(x^2+y^2+1)^2} & \dfrac{2x^2-2y^2+2}{(x^2+y^2+1)^2} \end{pmatrix}$$

~~Theorem~~

**Defn:** A stationary pt for $f$ is a point where $f'_{x_1} = f'_{x_2} = \dots = f'_{x_n} = 0$ FOC

**Thm:** (First order condition)

$\underline{x}^*$ local max/min $\Longrightarrow$ $\underline{x}^*$ stationary pt.

**Ex:** $f = x^3 + xy - y^3$

Stationary pts:

$$f'_x = 3x^2 + y = 0$$
$$f'_y = x - 3y^2 = 0$$

$y = -3x^2$: $x - 3\cdot(-3x^2)^2 = 0$

$x - 3\cdot 9\cdot x^4 = 0$

$x\cdot(1 - 27x^3) = 0$

$x = 0$ or $27x^3 = 1$

$x = 0 \Downarrow y = 0$

$x^3 = 1/27$

$x = 1/3 \Downarrow y = -3\cdot\left(\frac{1}{3}\right)^2 = -1/3$

Stationary pts:

$(x,y) = (0,0)$, $(1/3, -1/3)$

$f(0,0) = 0$ $\quad f(1/3, -1/3) = \dfrac{1-3+1}{27} = -\dfrac{1}{27}$

BI

Finding stationary pts for $f$ gives us candidates for local max/min, and therefore candidates for max/min

## Second order condition (SOC)

Assume that $\underline{x}^*$ is a stationary point for $f(\underline{x})$.

If $H(f)(\underline{x}^*)$ is positive defn. then $\underline{x}^*$ is local min

— '' — negative defn then $\underline{x}^*$ is local max

— '' — indefinite then $\underline{x}^*$ is a saddle point

(i.e. a stationary pt that is neither local max nor local min)

Test not conclusive if $H(f)(\underline{x}^*)$ is pos./neg. semidefn. (but not defn.)

Ex: $f = x^3 + xy - y^3$ $\quad H(f) = \begin{pmatrix} 6x & 1 \\ 1 & -6y \end{pmatrix}$

Stat. pts: $(0,0)$

$(1/3, -1/3)$

Classification:

$H(f)(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad D_1 = 0$, $D_2 = -1$ $\rightarrow$ indefinite $\Downarrow$ $(0,0)$ saddle pt

$H(f)(1/3,-1/3) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $\quad D_1 = 2$, $D_2 = 3$. positive defn. $\Downarrow$ $(1/3,-1/3)$ local min

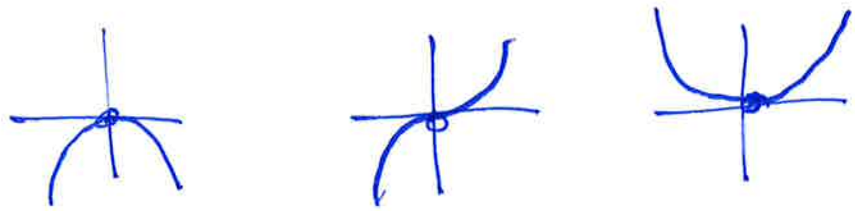
$f(-2,0) = -8$
no min

$f$ has no max

$f$ has local min in $(1/3,-1/3)$ $\boxed{f = -\frac{1}{27}}$

(candidate for min)

| | | | |
|---|---|---|---|
| **Ex:** | $f = -x^4$ | $f = x^3$ | $f = x^4$ |
| | $f'_x = -4x^3$ | $f' = 3x^2$ | $f' = 4x^3$ |
| **St. pts:** | $x = 0$ | $x = 0$ | $x = 0$ |
| **SOC:** | $H(f)(0) = -12x^2 = \underline{0}$ | $H(f)(0) = 6x = \underline{0}$ | $H(f)(0) = 12x^2 = \underline{0}$ |



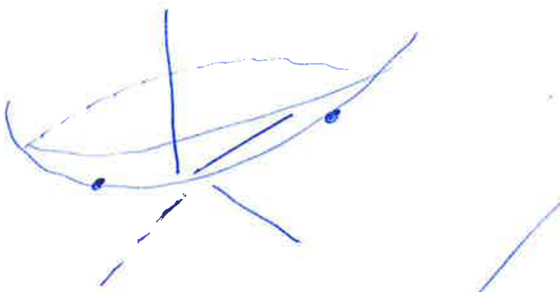
max — saddle pt. — min

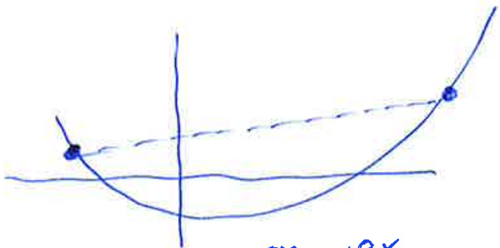
## ③ Convex and concave functions

$f(\underline{x})$ : function

<u>Defn:</u>

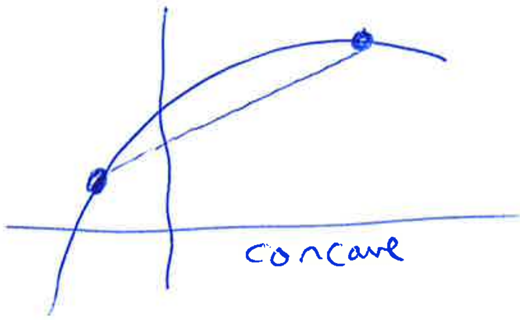
f is <u>convex</u> if for any two points on the graph of f, the straight line between these two pts lies <u>over</u> (or on) the graph.





convex

f is <u>concave</u> if for any two points on the graph of f, the straight line between these two pts lies <u>under</u> (or on) the graph



concave

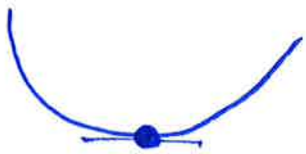
neither convex nor concave

<u>note</u>: convex / concave are global notions

**Thm:**

If $f$ is convex, then any stationary pt of $f$ is global min.

If $f$ is concave, then any stationary pt of $f$ is global max.



Convex

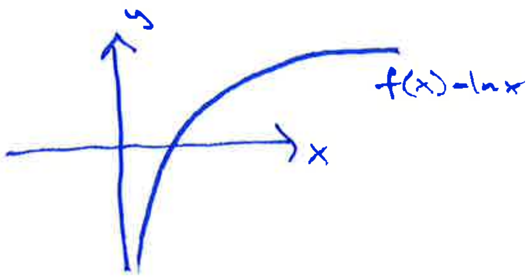
Concave

**Ex:** $f(x) = \ln x$, $x > 0$

$f'(x) = 1/x$

$f''(x) = -1/x^2 < 0$

$f$ concave




no stationary pts

no global max

## Criterion for convexity / concavity:

If $H(f)(\underline{x})$ is positive semidefinite for all $\underline{x}$, then $f$ is convex

If $H(f)(\underline{x})$ is negative semidefinite for all $\underline{x}$, then $f$ is concave

BI

$$\boxed{\begin{array}{lll} f \text{ convex} & \Longleftrightarrow & H(f)(\underline{x}) \text{ pos. semidefn. for all } \underline{x} \\ f \text{ concave} & \Longleftrightarrow & H(f)(\underline{x}) \text{ neg. semidefn. for all } x. \end{array}}$$

<u>Ex:</u> $f = x^3 + xy - y^3$ $\quad H(f) = \begin{pmatrix} 6x & 1 \\ 1 & -6y \end{pmatrix}$

$D_1 = 6x \qquad \Delta_1 = -6y$

$D_2 = -36xy - 1$

<u>convex:</u> $D_1 \geq 0 \quad \Delta_1 \geq 0$

$D_2 \geq 0$

$$\boxed{\begin{array}{l} 6x \geq 0 \quad -6y \geq 0 \\ -36xy - 1 \geq 0 \\ \text{for all } x, y \end{array}}$$

$D_1 = 6x$ can be pos. or neg. $\Big\} \Rightarrow$ <u>no</u> not convex

<u>concave:</u> $D_1 \leq 0 \quad 6x \leq 0$ for all $x, y$

<u>no</u> not concave

<u>Ex:</u> $f = e^{2x-y} \qquad f'_x = e^{2x-y} \cdot 2 \qquad f'_y = e^{2x-y} \cdot (-1)$

$$H(f) = \begin{pmatrix} 4e^{2x-y} & -2e^{2x-y} \\ -2e^{2x-y} & e^{2x-y} \end{pmatrix}$$

$D_1 = 4e^{2x-y} > 0$

$D_2 = 4 \cdot (e^{\cdot\cdot})^2 - 4(e^{\cdot\cdot})^2 = 0$

$\Delta_1^{22} = e^{2x-y} > 0$

$\Big\}$ pos. semidefn.

$\Downarrow$

$f$ <u>convex</u>

# An Example

Let $f(x,y) = x^2y^3 + y^2 - 2y$. Then $f$ has just one stationary pt, which is a local min. However, $f$ has no global min.

$$\left.\begin{aligned} f'_x &= 2xy^3 = 0 \\ f'_y &= 3x^2y^2 + 2y - 2 = 0 \end{aligned}\right\}$$

FOC: $x=0$ or $y=0$

If $x=0$, then (2) gives

$2y - 2 = 0 \Rightarrow y = 1$

If $y=0$, then (2) gives

$-2 = 0$ no solution

$$H(f) = \begin{pmatrix} 2y^3 & 6xy^2 \\ 6xy^2 & 6x^2y+2 \end{pmatrix}$$

$$H(f)(0,1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \begin{aligned} D_1 &= 2 > 0 \\ D_2 &= 4 > 0 \end{aligned}$$

pos. defn. $\Rightarrow$ $(0,1)$ local min

$$\boxed{\text{Conclusion: } (0,1) \text{ is the only stationary pt, with } f(0,1) = -1}$$

But $(x,y) = (0,1)$ with $f(0,1) = -1$ is not global min

For example,

$$f(3,-1) = -9 + 1 + 2 = -6 < -1$$

$$\boxed{\text{Conclusion: } f \text{ has no global min even if } (0,1) \text{ is local min and this is the only stationary point.}}$$