LECTURE 5

GRA 6035 MATHEMATICS

BI

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Plan:

1. Diagonalization and Markov chains
2. Quadratic forms, definiteness and principle minors.

Reading:

[MEJ] 6.2 (Ex. 3), 23.1 (Ex 23.4), 23.6, 13.1 - 13.5, 16.1 - 16.4, 23.8

1. Diagonalization (review)

A $n \times n$-matrix is diagonalizable if there is an invertible matrix P s.t. $P^{-1}AP = D$ is diagonal.

$P^{-1}AP = D \iff AP = PD \iff$

i) $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ has diagonal entries $\lambda_1, \dots, \lambda_n$, the eigenvalues, and each eigenvalue appears a number of times equal to its multiplicity.

ii) $P = (\underline{v_1} | \dots | \underline{v_n})$, where $\underline{v_i}$ is an eigenvector with eigenvalue λ_i, and $\{\underline{v_1}, \dots, \underline{v_n}\}$ are linearly independent. This means that the number of vectors among $\underline{v_1} \dots \underline{v_n}$ that has eigenvalue λ, is equal to the multiplicity m of λ.

Ex: Solution $\underline{v}$ given by two free variables s, t, then

$\underline{v} = s \cdot \underline{v_1} + t \cdot \underline{v_2}$

When λ is an eigenvalue of multiplicity m, we find the eigenvectors in E_λ by solving

$(A - \lambda I)\underline{v} = \underline{0}$

The number of degrees of freedom in this system is at least 1 and at most m.

Results:

a) If A is symmetric, it is diagonalizable.

b) If A has n distinct eigenvalues, it is diagonalizable.

In other cases, we must compute eigenvalues and eigenvectors, and check conditions i) - ii):

i) A has n eigenvalues, counted with multiplicity

ii) A has n linearly independent eigenvectors, or in other words, for each eigenvalue λ of multiplicity m, $(A-\lambda I)\underline{v} = \underline{0}$ has m degrees of freedom.

Applications: Markov chains

Ex: employed / unemployed

State vector: $\underline{x}_t = \begin{pmatrix} e_t \\ u_t \end{pmatrix}$ ← share of the pop. that is employed at time t.

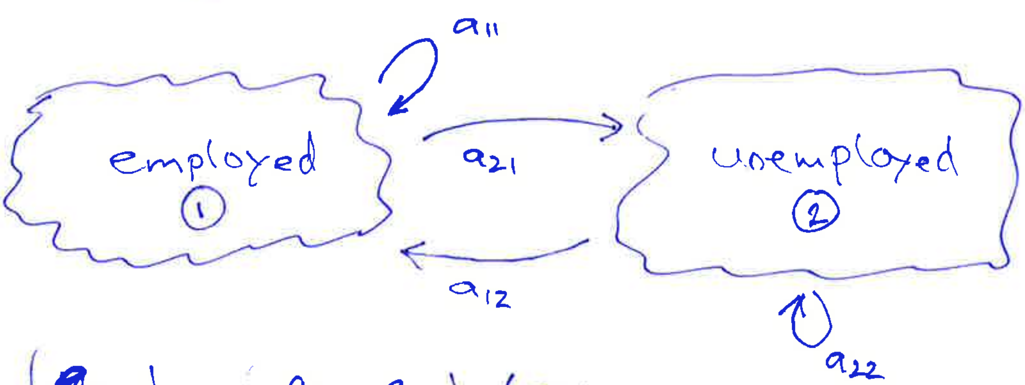
" unemployed "

transition matrix: A

2×2-matrix

$$\boxed{\underline{x}_{t+1} = A \cdot \underline{x}_t}$$

constant $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$



$$\begin{pmatrix} e_{t+1} \\ u_{t+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} e_t \\ u_t \end{pmatrix} = \begin{matrix} a_{11} e_t + a_{12} u_t \\ a_{21} e_t + a_{22} u_t \end{matrix}$$

What happens in the long run?

$$\underline{x}_0 \rightsquigarrow A\cdot \underline{x}_0 = \underline{x}_1 \rightsquigarrow A\cdot \underline{x}_1 = \underline{x}_2 \rightsquigarrow \dots \rightarrow \underline{x}_t = A^t\cdot \underline{x}_0$$

$$A\cdot \underline{x}_1 = A^2\cdot \underline{x}_0$$

Ex:

$$A = \begin{pmatrix} 0.98 & 0.136 \\ 0.02 & 0.864 \end{pmatrix} :$$ What is A^n when $n >> 0$?

(weekly data, US labour market)

Eigenvalues:

$$\begin{vmatrix} 0.98-\lambda & 0.136 \\ 0.02 & 0.864-\lambda \end{vmatrix} = \lambda^2 - 1.844\lambda + 0.844 = 0$$

$$\underline{\lambda = 1}, \quad \lambda = \underline{0.844}$$

Eigenvectors:

$\underline{\lambda = 1}$:

$$\begin{pmatrix} -0.02 & 0.136 \\ \cancel{0.02} & \cancel{-0.136} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad 0.02x = 0.136y$$

$$x = \frac{13.6}{2}y = 6.8y$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6.8y \\ y \end{pmatrix} = y\cdot\begin{pmatrix} 6.8 \\ 1 \end{pmatrix} \qquad 6.8y + y = 1 \Rightarrow y = \frac{1}{6.8+1}$$

$$y = \frac{1}{7.8} \Rightarrow \underline{v}_1 = \begin{pmatrix} 68/78 \\ 10/78 \end{pmatrix}$$

$$A\cdot\begin{pmatrix} 68/78 \\ 10/78 \end{pmatrix} = \underbrace{\begin{pmatrix} 68/78 \\ 10/78 \end{pmatrix}}_{= \underline{v}_1} : \text{ Equilibrium state } \underline{v}_1 = \underline{\underline{\begin{pmatrix} 68/78 \\ 10/78 \end{pmatrix}}}$$

$$\underline{x}_0 = \underline{v}_1 \rightsquigarrow \underline{x}_t = A^t\cdot\underline{v}_1 = \underline{v}_1 \leftarrow \lambda_1 = 1$$

$\underline{\lambda = 0.844}$:

$$\begin{pmatrix} x \\ y \end{pmatrix} = y\cdot\begin{pmatrix} 1 \\ -1 \end{pmatrix} = y\cdot\underline{v}_2 \qquad \lambda_2 = 0.844$$

$$\underline{x}_0 = \underline{v}_2 : \quad x_1 = A\cdot\underline{v}_2 = 0.844\cdot\underline{v}_2 \rightarrow \dots \rightarrow \underline{x}_t = A^t\cdot\underline{x}_0 = \underbrace{0.844^t}_{\to 0 \text{ as } t\to\infty}\cdot\underline{v}_2$$

$$\underline{x_0} = \begin{pmatrix} 0.90 \\ 0.10 \end{pmatrix} : \quad \underline{x_0} = \overset{(c_1 = 1)}{c_1} \cdot \underline{v_1} + \overset{(c_2 = 11/390)}{c_2} \cdot \underline{v_2}$$

Since A is diagonalizable, i.e. we have n lin. independent eigenvectors

$$x_0 = c_1 \underline{v_1} + c_2 \underline{v_2} : \quad x_1 = A \cdot \underline{x_0} = A \cdot (c_1 \underline{v_1} + c_2 \underline{v_2})$$

$$= c_1 \cdot A\underline{v_1} + c_2 \cdot A\underline{v_2}$$

$$= c_1 \cdot \underline{v_1} + c_2 \cdot 0.844 \, \underline{v_2}$$

$\vdots$

$$x_t = A^t \cdot x_0 = c_1 \cdot \underline{v_1} + c_2 \cdot 0.844^t \cdot \underline{v_2}$$

$$\lim_{t \to \infty} \underline{x_t} = \underline{c_1 \cdot v_1} = \underline{v_1} = \underline{\begin{pmatrix} 68/78 \\ 10/78 \end{pmatrix}}$$

unique state vector which is eigenvector for $\underline{\lambda = 1}$.

$$\underline{v_1} \simeq \begin{pmatrix} 0.872 \\ 0.128 \end{pmatrix}$$

Results for Markov chains

General transition matrix A: $\begin{cases} * & 0 \le a_{ij} \le 1 \\ * & \text{each column has sum} = 1. \end{cases}$

It is regular if $a_{ij} > 0$ for all i,j. Then we have:

i) $\lambda = 1$ is an eigenvalue, and it is dominant (all other eigenvalues λ_i has $|\lambda_i| < 1$)

ii) There is a unique state vector $\underline{v}$ that is an eigenvector with $\lambda = 1$.

iii) For any initial state $\underline{x_0}$, we have that

$$\lim_{t \to \infty} \underline{x_t} = \lim_{t \to \infty} A^t \cdot \underline{x_0} = \underline{v}$$

Markov process

Ex: Families are classified as U (urban), S (suburban) and R (rural). At time $\underline{t=n}$ (after n years), the share of families in these groups can be described by the $\underline{\text{state vector}}$

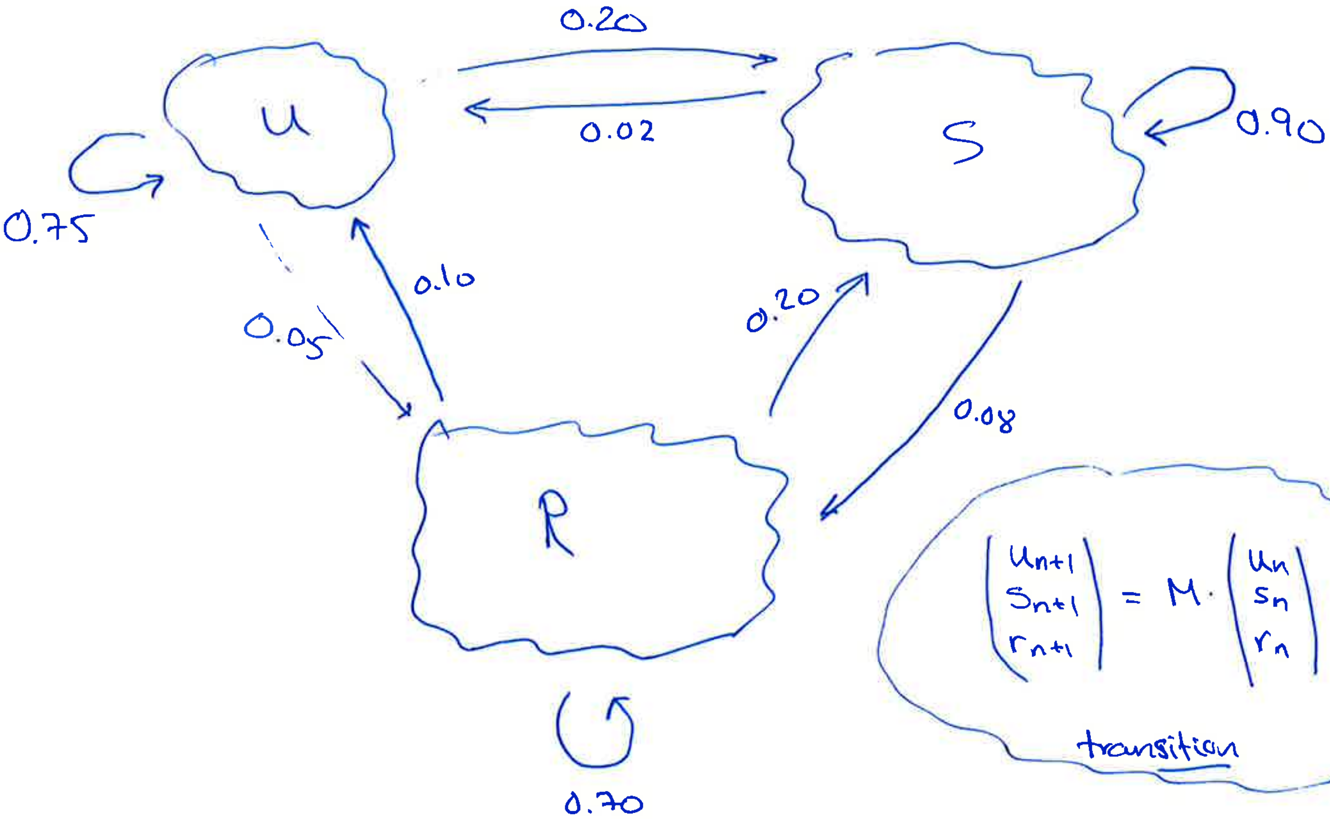
$$\underline{V}_n = \begin{pmatrix} u_n \\ s_n \\ r_n \end{pmatrix} \qquad \begin{cases} u_n \geq 0 \\ s_n \geq 0 \\ r_n \geq 0 \end{cases}, \quad u_n + s_n + r_n = 1$$

Ex: $\underline{V} = \begin{pmatrix} 0.8 \\ 0.1 \\ 0.1 \end{pmatrix}$

From year n to year $n+1$, the change in the shares are given by a transition matrix or Markov matrix

$$M = \begin{pmatrix} 0.75 & 0.02 & 0.10 \\ 0.20 & 0.90 & 0.20 \\ 0.05 & 0.08 & 0.70 \end{pmatrix} \qquad \begin{cases} m_{ij} \geq 0 \\ \text{each column has sum } 1 \end{cases}$$

It can be described graphically as follows:



$$\begin{pmatrix} u_{n+1} \\ s_{n+1} \\ r_{n+1} \end{pmatrix} = M \cdot \begin{pmatrix} u_n \\ s_n \\ r_n \end{pmatrix}$$

transition

Markov process:

$$\underline{V}_0 = \begin{pmatrix} u_0 \\ s_0 \\ r_0 \end{pmatrix} \dashrightarrow \underline{V}_1 = M \cdot \underline{V}_0 \dashrightarrow \underline{V}_2 = M\underline{V}_1 = M^2 \underline{V}_0 \dashrightarrow \cdots \longrightarrow \underline{V}_n = M^n \cdot \underline{V}_0$$

start

The Markov process is <u>regular</u> if $m_{ij} > 0$ for all i,j. We assume that this the case. The following holds for all <u>regular Markov processes</u>:

<u>Fact:</u> i) $\lambda = 1$ is an eigenvalue of M, and there is a unique eigenvector $\underline{v}$ with eigenvalue $\lambda = 1$ that is a state vector (i.e. $\underline{v} = (v_i)$ with $v_i \geq 0$, $v_1 + \dots + v_k = 1$)

ii) $\lim_{n \to \infty} M^n \cdot \underline{v_0} = \underline{v}$ and $\lim_{n \to \infty} M^n = (\underline{v} | \underline{v} | \dots | \underline{v})$

<u>Ex:</u> $M = \begin{pmatrix} 0.75 & 0.02 & 0.10 \\ 0.20 & 0.90 & 0.20 \\ 0.05 & 0.08 & 0.70 \end{pmatrix}$ $\qquad D = \begin{pmatrix} 1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$

<u>$\lambda = 1$:</u> $\begin{pmatrix} -0.25 & 0.02 & 0.10 \\ 0.20 & -0.10 & 0.20 \\ 0.05 & 0.08 & -0.30 \end{pmatrix} \to \begin{pmatrix} 5 & 8 & -30 \\ -25 & 2 & 10 \\ 20 & -10 & 20 \end{pmatrix} \to \begin{pmatrix} 5 & 8 & -30 \\ 0 & 42 & -140 \\ 0 & -42 & 140 \end{pmatrix}$

$\to \begin{pmatrix} 5 & 8 & -30 \\ 0 & 42 & -140 \\ 0 & 0 & 0 \end{pmatrix}$

$5x + 8y - 30z = 0$

$42y - 140z = 0$

z free

$y = \frac{140z}{42} = \frac{10}{3}z$

$5x = 30 \cdot z - 8 \cdot \frac{10}{3}z = \frac{90-80}{3}z$

$x = \frac{2}{3}z$

$\frac{2}{3}z + \frac{10}{3}z + z = 1$

$5z = 1$

$z = 1/5$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2/3 \cdot z \\ 10/3 \cdot z \\ z \end{pmatrix} = z/3 \cdot \begin{pmatrix} 2 \\ 10 \\ 3 \end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix} 2/15 \\ 10/15 \\ 3/15 \end{pmatrix}$ (with $z = 1/5$)

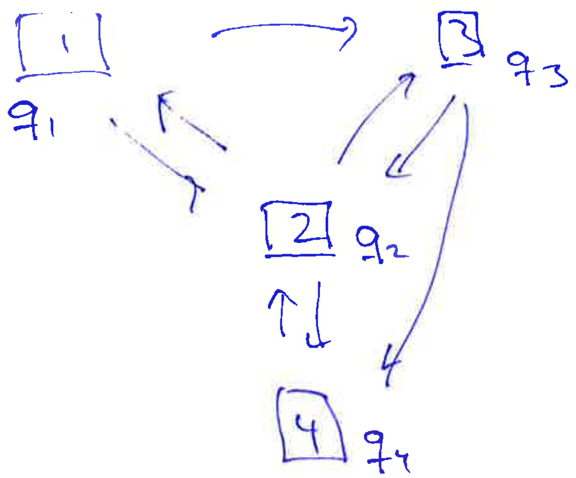
<u>Conclusion:</u> As $n \to \infty$ (in the long run) $u = 2/15 \approx 13.3\%$ of families are urban, $s = 10/15 \approx 66.7\%$ are suburban, and $r = 3/15 = 20\%$ are rural.

<u>Check:</u> Compute M^{10}, M^{50}, M^{100} using Wolfram Alpha or other software.

It is also possible to compute M^n as

$$M^n = P \cdot \begin{pmatrix} \overset{1}{\overset{\shortparallel}{\lambda_1}} & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}^n \cdot P^{-1} \underset{\substack{\uparrow \\ \text{since } \lambda_1 = 1,\ \lambda_2, \lambda_3 < 1}}{\simeq} P \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} P^{-1}$$

Ex: Google's Pagerank algorithm



$q_1 = K \cdot (q_2)$

$q_2 = K \cdot (q_1 + q_3 + q_4)$

$q_3 = K \cdot (q_1 + q_2)$

$q_4 = K \cdot (q_2 + q_3)$

$$\underline{q} = K \cdot \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \underline{q}$$

$$\underline{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}$$

$$\lambda \cdot \underline{q} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \underline{q}$$

replace with M

$$M = \begin{pmatrix} 0 & 1/3 & 0 & 0 \\ 1/2 & 0 & 1/2 & 1 \\ 1/2 & 1/3 & 0 & 0 \\ 0 & 1/3 & 1/2 & 0 \end{pmatrix}$$

Eigenvectors for M with $\underline{\lambda = 1}$,

$$\underline{V} = \begin{pmatrix} 4/29 \\ 12/29 \\ 6/29 \\ 7/29 \end{pmatrix}$$

$q_1 = 4/29$

$q_2 = 12/29$

$q_3 = 6/29$

$q_4 = 7/29$

$$M^n \longrightarrow (\underline{V} | \underline{V} | \ldots | \underline{V})$$

as $n \to \infty$

efficient computation

BI

(2) Quadratic forms.

Defn: A quadratic form in $x_1, x_2, \dots, x_n$ has the form

$$f(x_1,\dots,x_n) = c_{11}x_1^2 + c_{12}x_1x_2 + c_{13}x_1x_3 + \dots + c_{1n}x_1x_n$$
$$+ \quad c_{22}x_2^2 + c_{23}x_2x_3 + \dots$$
$$\ddots$$
$$+ \; c_{nn}x_n^2$$

Ex: $f(x,y) = ax^2 + bxy + cy^2$

$$f(x_1,x_2,x_3) = c_{11}x_1^2 + c_{12}x_1x_2 + c_{13}x_1x_3 + c_{22}x_2^2 + c_{23}x_2x_3 + c_{33}x_3^2$$

Matrix form:

Ex: $2x^2 + 4xy - 6y^2 = (x \;\; y)\underbrace{\begin{pmatrix} 2 & 2 \\ 2 & -6 \end{pmatrix}}_{=A}\begin{pmatrix} x \\ y \end{pmatrix}$

Check:

$$(x \;\; y)\cdot\begin{pmatrix} 2 & 2 \\ 2 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+2y & 2x-6y \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$
$$= (2x+2y)x + (2x-6y)y$$
$$= \underset{a_{11}}{\underset{\parallel}{2}}x^2 + \underset{a_{21}}{\underset{\parallel}{2}}yx + \underset{a_{12}}{\underset{\parallel}{2}}xy \underset{a_{22}}{\underset{\parallel}{-6}}y^2$$
$$= a_{11}x^2 + (a_{12}+a_{21})xy + a_{22}y^2$$

Fact: If $f(\underline{x}) = f(x_1 \dots x_n)$ is quadratic form, then there is a unique symmetric $n\times n$-matrix A s.t.

$$f(\underline{x}) = (x_1 \dots x_n) \cdot A \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \underline{x}^T \cdot A \cdot \underline{x}$$

$$\underline{x} = (x_1, x_2, \dots, x_n) \approx \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \underline{x}$$

Ex: $f(x_1, x_2, x_3) = \underline{3x_1^2} + 2x_1x_3 \underline{-x_2^2} + \underline{x_3^2}$

$$f(\underline{x}) = (x_1 \; x_2 \; x_3) \cdot \begin{pmatrix} 3 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= \underline{x}^T \cdot A \cdot \underline{x}$$

Definiteness: f quadratic form, $f(\underline{0}) = 0$

f **pos. definite** if $f(\underline{x}) > 0$ for all $\underline{x} \neq \underline{0}$

f **positive semidefinite** if $f(\underline{x}) \geq 0$ for all $\underline{x}$

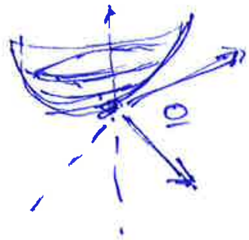
f **neg. definite** if $f(\underline{x}) < 0$ for all $\underline{x} \neq \underline{0}$

f **negative semidefinite** if $f(\underline{x}) \leq 0$ for all $\underline{x}$

f **indefinite** otherwise, i.e. f is neither positive nor negative semidef.

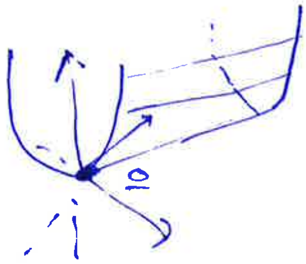
Interpretation: Graph of f

f pos. defn.:



$\underline{x} = \underline{0}$ global min.

f pos. semidefn:

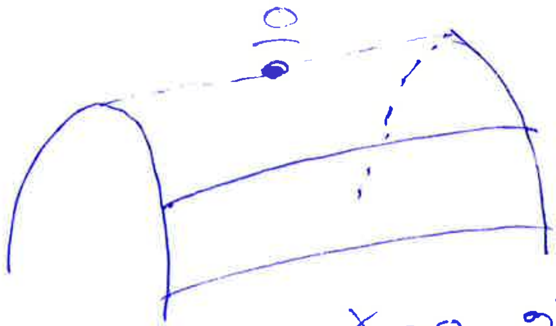


$\underline{x} = \underline{0}$ global min (and some other pts)

f neg. defn

$\underline{x} = \underline{0}$ global max.

f neg. semidefn.



$\underline{x} = \underline{0}$ global max (and some other pts).

f indefinite $\underline{x} = \underline{0}$ saddle pt.

Ex: $f(x_1,x_2,x_3) = 2x_1^2 + x_2^2 + x_3^2$ pos. definite

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 7 \end{pmatrix}$ $\quad f(x_1,x_2,x_3) = 2x_1^2 + 7x_3^2$ pos. semi-definite

$f(\underline{x}) \geq 0$

$f(0,1,0) = 0$

not pos. definite

$f(x_1,x_2,x_3) = x_1^2 - x_2^2 + x_3^2$

indefinite

$f(1,0,0) = 1 > 0$

$f(0,1,0) = -1 < 0$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Result: $f(\underline{x})$ quadratic form with symm. matrix A and eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of A.

f pos defn. $\iff \lambda_1, \lambda_2, \dots, \lambda_n > 0$

f pos. semidefn. $\iff \lambda_1, \lambda_2, \dots, \lambda_n \geq 0$

f neg. defn. $\iff \lambda_1, \lambda_2, \dots, \lambda_n < 0$

f neg semidefn. $\iff \lambda_1, \lambda_2, \dots, \lambda_n \leq 0$

f indefinite $\iff$ A has both positive and negative eigenvalues

Since A is symmetric $\Rightarrow$ A diagonalizable, there is a change of variables $u_1, \dots, u_n$ s.t.

$$f = \lambda_1 u_1^2 + \lambda_2 \cdot u_2^2 + \dots + \lambda_n \cdot u_n^2$$

Ex: $f(x,y) = 2x^2 + 2xy + 2y^2$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Eigenvalues: $\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = 0$

$\underline{\lambda = 1}, \underline{\lambda = 3}$

f and A are <u>positive defn</u> since $\lambda_1 > 0, \lambda_2 > 0$.

Ex: $f(x,y,z) = x^2 + 4xy + 8xz + 3y^2 - 2yz + 2z^2$

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & -1 \\ 4 & -1 & 2 \end{pmatrix}$$

<u>Eigenvalues:</u>

$$\begin{vmatrix} 1-\lambda & 2 & 4 \\ 2 & 3-\lambda & -1 \\ 4 & -1 & 2-\lambda \end{vmatrix} = (1-\lambda)\cdot\left((3-\lambda)(2-\lambda) - 1\right) - 2\left(2(2-\lambda)+4\right) + 4\left(-2-4(3-\lambda)\right)$$

$$= (1-\lambda)(3-\lambda)(2-\lambda) - (1-\lambda) - 2(8-2\lambda) + 4(4\lambda-14)$$

$$= 0 \qquad \text{(difficult equation!)}$$

$$\left.\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 &= \operatorname{tr} A = 6 \\ \lambda_1 \cdot \lambda_2 \cdot \lambda_3 &= |A| = -67 \end{aligned}\right\} \text{one negative and two positive eigenvalues}$$

$\Downarrow$

<u>indefinite</u>

Principal minors

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & -1 \\ 4 & -1 & 2 \end{pmatrix}$$

$$f = x_1^2 + 4x_1x_2 + 8x_1x_3 + 3x_2^2 - 2x_2x_3 + 2x_3^2$$

Leading principle minors:

$D_1 = 1 > 0$

$D_2 = -1 < 0$

$D_3 = 2 \cdot D_2 + 1 \cdot (-9) + 4 \cdot (-14)$
$= -2 - 9 - 56 = -67 < 0$

A indefinite

from result below.

Principle minor: Minor computed using the same rows as columns

Δ_i : principle minor of order i

Leading principle minor: D_i = minor computed using rows $1, 2, \ldots, i$ and col's $1, 2, \ldots, i$

Result:

$D_1, D_2, \ldots, D_n > 0 \iff$ A pos. definite

$D_1 < 0, D_2 > 0, D_3 < 0, \ldots \iff$ A neg. definite

$\Delta_1, \Delta_2, \ldots, \Delta_n \geq 0 \iff$ A positive semidefinite

$\Delta_1 \leq 0, \Delta_2 \geq 0, \ldots \iff$ A negative semidefinite

otherwise A indefinite

Complete list of principle minors:

Δ_1: $\Delta_1^{11} = \underline{1}$ $\quad \Delta_1^{22} = \underline{3}$ $\quad \Delta_1^{33} = \underline{2}$

row 1
col 1

row 1,2
cols 1,2

Δ_2: $\Delta_2^{12,12} = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = \underline{-1}$

$\Delta_2^{13,13} = \begin{vmatrix} 1 & 4 \\ 4 & 2 \end{vmatrix} = \underline{-14}$

$\Delta_2^{23,23} = \begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix} = \underline{5}$

Δ_3: $\Delta_3^{123,123} = |A| = \dots = \underline{-67}$

all principle minors

$D_1 = \Delta_1^{11} = \underline{1}$

$D_2 = \Delta_2^{12} = \underline{-1}$

$D_3 = \Delta_3^{123,123}$
$= |A| = \underline{-67}$

leading principle minors (subset)

Ex: $x^2 + 2xy + y^2 + z^2 \longrightarrow A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$D_1 = 1$

$D_2 = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$

$D_3 = |A| = 1 \cdot D_2 = 0$

not pos. defn., but could be pos. semidefn. since $D_1, D_2, D_3 \geq 0$

check all principle minors:

$$\left.\begin{aligned} \Delta_1 &= 1, 1, 1 \geq 0 \\ \Delta_2 &= 0, 1, 1, \geq 0 \\ \Delta_3 &= 0 \geq 0 \end{aligned}\right\} \quad \Delta_i \geq 0 \Rightarrow \underline{\text{positive semi. defn.}}$$