LECTURE 4

EIVIND ERIKSEN   SEP 10, 2015

GRA 6035

MATHEMATICS

BI

Plan:

1. Eigenvalues and eigenvectors
2. Diagonalization

Reading:

[ME] 23.1-23.4, 23.6-23.7, 23.9

1 Eigenvalues and eigenvectors

A: $n \times n$-matrix

$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \rightsquigarrow A \cdot \underline{v}$

n-vector   n-vector

Definition:

A number λ is an eigenvalue for A if the equation

$$A \cdot \underline{v} = \lambda \cdot \underline{v} \quad (*)$$

has non-trivial solutions $\underline{v} \neq \underline{0}$. In that case, the $E_\lambda = \{ \underline{v} : A\underline{v} = \lambda \cdot \underline{v} \}$ are called the eigenvectors of A with eigenvalue λ. The collection E_λ of eigenvectors is called the eigenspace.

Ex:
$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$\underline{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$:
$$A \cdot \underline{v} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$
$$\lambda \cdot \underline{v} = \lambda \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda \\ \lambda \end{pmatrix} \quad /\!/ \quad \lambda = 3$$

$\lambda = 3$ eigenvalue, $\underline{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ eigenvector in E_3.

$\underline{w} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$: $\quad A \cdot \underline{w} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \neq \lambda \cdot \underline{w}$

not eigenvector

(*) $\quad A \cdot \underline{v} = \lambda \cdot \underline{v} \quad , \quad \underline{v} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \cdot \begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{aligned} 2x + y &= \lambda x \\ x + 2y &= \lambda y \end{aligned}$$

$$\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = A - \lambda I \qquad \begin{aligned} (2-\lambda)x + y &= 0 \\ x + (2-\lambda)y &= 0 \end{aligned}$$

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = |A - \lambda I|$$

$\neq 0$: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix}^{-1} \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ only the trivial solution

$= 0$: at least one free variable since $\operatorname{rk}(A - \lambda I) < 2$ inf. many solutions

Fact:

i) $A\underline{v} = \lambda \underline{v}$ (*) $\Longleftrightarrow$ $(A - \lambda I)\underline{v} = \underline{0}$ (**)

ii) λ eigenvalue $\Longleftrightarrow$ $|A - \lambda I| = 0$
characteristic equation

When we want to compute the eigenvalues of A, we solve $|A - \lambda I| = 0$.

Ex: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0 \qquad \text{(char. eqn.)}$$

$$(2-\lambda)\cdot(2-\lambda) - 1^2 = 0$$

$$4 - \cancel{4}\lambda + \lambda^2 - 1 = 0$$

$$\underline{\lambda^2 - 4\lambda + 3 = 0} \qquad \lambda_1 = \underline{3}\,,\ \lambda_2 = \underline{1}$$

$\underline{\lambda = 3:}$ Eigenvectors are solutions of

$$A\cdot \underline{v} = 3\underline{v} \quad (*)$$
$$\Updownarrow$$
$$(A - 3I)\underline{v} = \underline{0} \quad (**)$$

$$\begin{pmatrix} 2-3 & 1 \\ 1 & 2-3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ \cancel{1} & \cancel{-1} \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad -x + y = 0$$

$x = y$
y : free

Eigenvectors:

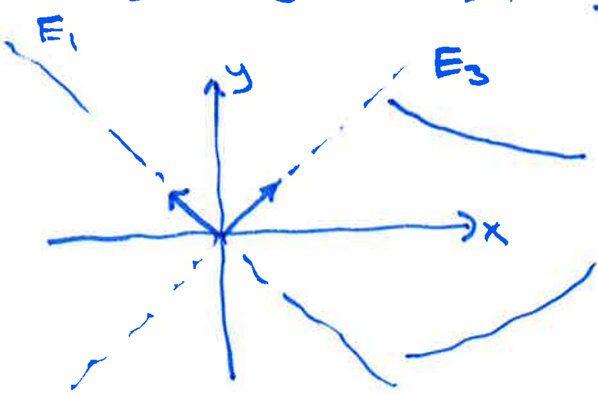
$$\underline{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = y\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \underline{E_3 = \left\{ y\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix} : y \right\}}$$

$\underline{\lambda = 1:}$

$$\begin{pmatrix} 2-1 & 1 \\ 1 & 2-1 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ \cancel{1} & \cancel{1} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad x + y = 0$$

y free

$$\underline{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = \underline{y\cdot\begin{pmatrix} -1 \\ 1 \end{pmatrix}} \qquad \underline{E_1 = \left\{ y\begin{pmatrix} -1 \\ 1 \end{pmatrix} : y \right\}}$$



$\underline{v} \in E_3$: $A\cdot\underline{v} = 3\underline{v}$

$\underline{v} \in E_1$: $A\cdot\underline{v} = \underline{v}$

General method: Finding eigenvalues and eigenvectors

A: $n\times n$-matrix $\rightsquigarrow$ $A-\lambda I = A - \begin{pmatrix} \lambda & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 0 & \cdots & 0 \\ \vdots & & \ddots & & \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}$

① Eigenvalues = Solution of $|A-\lambda I| = 0$.
(char. eqn.)

② For each eigenvalue λ found in ①, compute eigenvectors with eigenvalue λ by solving

(**) $(A-\lambda I)\cdot \underline{v} = \underline{0}$ for that specific λ.
(Linear system)

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} : \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

Char. eqn: $(-\lambda)^n + \ldots$ (lower degree terms) $= 0$

Case $n=2$:

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$: Char. eqn. $\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$

$$(a-\lambda)(d-\lambda) - bc = 0$$

$$ad - a\lambda - d\lambda + \lambda^2 - bc = 0$$

$$\lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

$$\boxed{\lambda^2 - tr(A)\cdot\lambda + det(A) = 0}$$

$tr(A) = a+d$

trace = sum of diagonal elements $= a_{11} + a_{22} + \cdots + a_{nn}$

Ex: $A = \begin{pmatrix} 7 & 1 \\ 1 & 7 \end{pmatrix}$

$$\lambda^2 - 14\lambda + 48 = 0$$

$$\lambda = \frac{14 \pm \sqrt{14^2 - 4 \cdot 48}}{2}$$

$$= 7 \pm \frac{\cancel{2}\sqrt{7^2 - 48}}{\cancel{2}} = 7 \pm 1$$

$$\lambda_1 = 8 \qquad \lambda = 6$$

Viete's formula:

If $\lambda^2 - tr(A) \cdot \lambda + det(A) = 0$ has roots λ_1 and λ_2, then

$$\boxed{\begin{aligned} \lambda_1 + \lambda_2 &= tr(A) \\ \lambda_1 \cdot \lambda_2 &= det(A) \end{aligned}}$$

General fact:

If A is an nxn-matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then

$$\lambda_1 + \lambda_2 + \ldots + \lambda_n = tr(A) = a_{11} + a_{22} + \ldots + a_{nn}$$

$$\lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n = det(A)$$

General fact:

If A is a symmetric nxn-matrix, then it has (real) eigenvalues.

Ex: $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\lambda^2 + 1 = 0$

$\lambda^2 = -1$

no (real) solutions

$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$\lambda^2 - 2\lambda + 1 = 0$

$(\lambda - 1)^2 = 0 \qquad (\lambda-1)\cdot(\lambda-1) = 0$

$\lambda_1 = 1 \quad \lambda_2 = 1$ (multiplicity 2)

Ex: $A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$\begin{vmatrix} 0-\lambda & -1 & 0 \\ 1 & 0-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$

$(3-\lambda) \cdot \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$

$(3-\lambda)\cdot(\lambda^2+1) = 0$

$\lambda = 3$ or $\lambda^2 + 1 = 0$

no solutions

Expected number of solutions: 3

In this case: 1 solution

$\lambda_1 = 3$ (mult. 1)

$\lambda = c$ has multiplicity m $\iff$ $(\lambda - c)^m$ is a factor in $|A - \lambda I|$.

Ex:

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

$$\begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 3-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0 \qquad \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$(3-\lambda)\cdot \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\cdot(\lambda^2-4\lambda+3)=0$$

$$\underline{\lambda_1=3} \qquad \lambda^2-4\lambda+3=0$$

$$\underbrace{\underline{\lambda_2=3}}_{\text{multiplicity } 2} \qquad \underbrace{\underline{\lambda_3=1}}_{\text{mult. } 1}$$

$$(3-\lambda)\cdot(\lambda-3)(\lambda-1)$$

Eigenvectors:

$\underline{\lambda=3}$:

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{matrix} -x+z=0 \\ y \text{ free} \\ z \text{ free} \end{matrix}$$

$$\underline{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} z \\ 0 \\ z \end{pmatrix} = \underline{y\cdot\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z\cdot\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}$$

$$= \underline{\text{span}\left(\underline{v_1}=\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \underline{v_2}=\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right)}$$

$\underline{\lambda=1}$:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ \cancel{1} & \cancel{0} & \cancel{1} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{matrix} x+z=0 & x=-z \\ 2y=0 & y=0 \\ \underline{z \text{ free}} & z=z \end{matrix}$$

$$\underline{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ 0 \\ z \end{pmatrix} = \underline{z\cdot\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}} \Rightarrow \text{span}\left(\underline{v_3}=\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right)$$

② Diagonalization

Defn: An $n \times n$-matrix A is diagonalizable if there is an invertible matrix P such that

$$P^{-1} \cdot A \cdot P = D, \text{ a diagonal matrix}$$

Result:

If A has n eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ counted with multiplicity, and n linearly independent eigenvectors $\underline{v_1}, \underline{v_2}, \dots, \underline{v_n}$, then A is diagonalizable and

$$P = \left(\underline{v_1} \;\vert\; \underline{v_2} \;\vert\; \dots \;\vert\; \underline{v_n} \right) \qquad D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

when $\underline{v_i}$ is an eigenvector with eigenvalue λ_i.

Ex: $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$

Eigenvalues: $\underbrace{\lambda_1 = \lambda_2 = 3}_{\text{mult. } 2}, \lambda_3 = 1$

Eigenvectors: $\underline{v_1}, \underline{v_2}, \underline{v_3}$

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad P = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$\lambda = 3$ $\quad \lambda = 1$

A is diagonalizable: $P^{-1}AP = D$

Why is $P^{-1}AP = D$ when P, D are chosen as stated? $P = (\underline{v_1}|\ldots|\underline{v_n})$, $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

BI

$$A \cdot P = A \cdot (\underline{v_1} | \underline{v_2} | \ldots | \underline{v_n}) = (A\underline{v_1} | A\underline{v_2} | \ldots)$$

$$= (\lambda_1 \underline{v_1} | \lambda_2 \underline{v_2} | \ldots \lambda_n \underline{v_n})$$

$$\cancel{= \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & & & \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix} \cdot (\underline{v_1} | \underline{v_2} | \ldots | \underline{v_n})}$$

$$\cancel{= D \cdot P}$$

$$= (\underline{v_1} | \underline{v_2} | \ldots | \underline{v_n}) \cdot \begin{pmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \ldots & \lambda_n \end{pmatrix}$$

$$= (\lambda_1 \underline{v_1} | \lambda_2 \underline{v_2} | \ldots | \lambda_n \underline{v_n})$$

$$= P \cdot D$$

$$A \cdot P = PD \implies \underline{P^{-1}AP = P^{-1}PD = D}$$

A <u>diagonalizable</u> (n×n-matrix) $\iff$ i) Enough eigenvalues: $\underline{\underline{n}}$ eigenvalues counted with multiplicity

and

ii) Enough eigenvectors: n lin. independent eigenvectors

$\Updownarrow$

For each eigenvalue λ, we have

free vars = m (multiplicity of λ)

Some useful facts:

i) When λ is an eigenvalue of with multiplicity m, then the linear system

$$(A-\lambda I)\underline{v} = 0$$

has at least one free variable, and <u>at most m</u> free variables.

ii) If A is symmetric, then A has n eigenvalues $\lambda_1, \dots, \lambda_n$ counted with multiplicity, and for each eigenvalue λ with multiplicity m,

$(A-\lambda I)\underline{v} = 0$ has $\underline{m}$ free variables.

This means:

If A is symmetric, then A is diagonalizable