# LECTURE 3

EIVIND ERIKSEN, SEP 03 2015

GRA 6035 **BI**

MATHEMATICS

**Plan:**

1. Vectors
2. Linear independence
3. Rank and linear independence

**Reading:**

[ME] 10.1-10.3, (10.4-10.7), 11.1

**Revision:**

$$\left\{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{array}\right. \implies \left\{\begin{array}{l} A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \text{ coeff. matrix} \\ \underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \\ \hat{A} = \left( \begin{array}{c|c} A & \underline{b} \end{array} \right) \text{ augmented matrix} \end{array}\right.$$

linear system, $m \times n$

**Fact:** * number of solutions given by pivot positions:

i) $rk\, A < rk\, \hat{A}$ $\longleftrightarrow$ pivot position in last col. of $\hat{A}$ $\longleftrightarrow$ inconsistent (no solutions)

ii) $rk\, A = rk\, \hat{A}$ $\longleftrightarrow$ no pivot pos. in last col. of $\hat{A}$ $\longleftrightarrow$ consistent (solutions)

In case ii), there are $\boxed{n - rk(A)}$ degrees of freedom

(and $n - rk(A)$ variable col's without a pivot position)

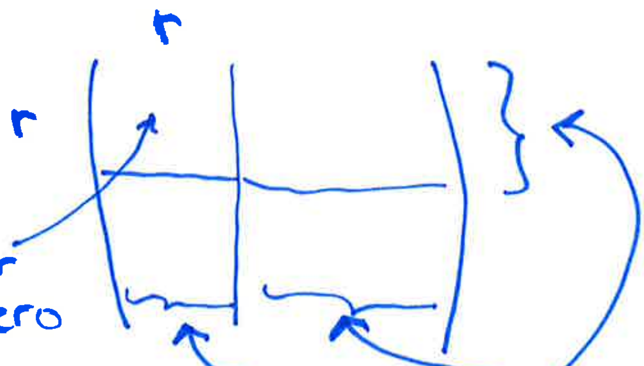
* pivot positions can be found using Gaussian elimination or <u>minors</u>. With minors:

  * rank = max. order of non-zero minor
  * if $rk\,A = rk\,\hat{A}$ (consistent system), and $rk\,A = r$

    since $M_{\underbrace{i_1 i_2 \ldots i_r}_{\text{rows}},\ \underbrace{j_1 j_2 \ldots j_r}_{\text{cols}}} \neq 0$, then we can

    solve eqn. $(i_1), (i_2), \ldots, (i_r)$ for $x_{j_1}, x_{j_2}, \ldots, x_{j_r}$

    and the remaining variables are free.




this minor is non-zero

$\Downarrow$

use these <u>equations</u>
to <u>solve for these</u> variables
and
these variables are <u>free</u>

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# 1 Vectors (col. vector)

An $m$-vector is an $m \times 1$-matrix, and we write

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$$

## Computing with vectors:

- we can add/subtract $m$-vectors, and compute scalar multiples of $m$-vectors

Ex:

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix} \qquad 4 \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$$

- linear combination of vectors

If $\underline{v_1}, \underline{v_2}, \ldots, \underline{v_n}$ are $m$-vectors, then a linear combination of $\{\underline{v_1}, \ldots, \underline{v_n}\}$ is an expression of the form

$$c_1 \cdot \underline{v_1} + c_2 \cdot \underline{v_2} + c_3 \cdot \underline{v_3} + \ldots + c_n \cdot \underline{v_n}$$

where $c_1, c_2, \ldots, c_n$ are given numbers.

Ex: $\underline{v_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\underline{v_2} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

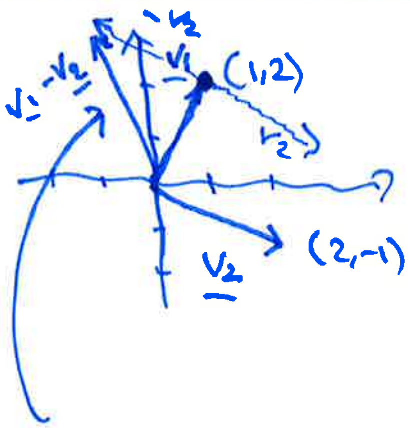
Linear comb: $c_1 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} c_1 \\ 2c_1 \end{pmatrix} + \begin{pmatrix} 2c_2 \\ -c_2 \end{pmatrix}$

$$= \begin{pmatrix} c_1 + 2c_2 \\ 2c_1 - c_2 \end{pmatrix}$$

Ex: $c_1 = 1, c_2 = -1$: $1 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

$\underline{v_1} - \underline{v_2} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

## Geometric interpretation:

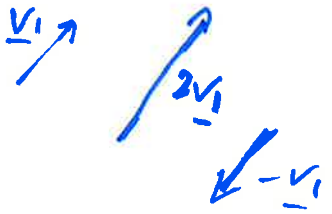



$\underline{v_1} - \underline{v_2}$

$\underline{v_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \leftrightarrow$ arrow from (0,0) to (1,2)

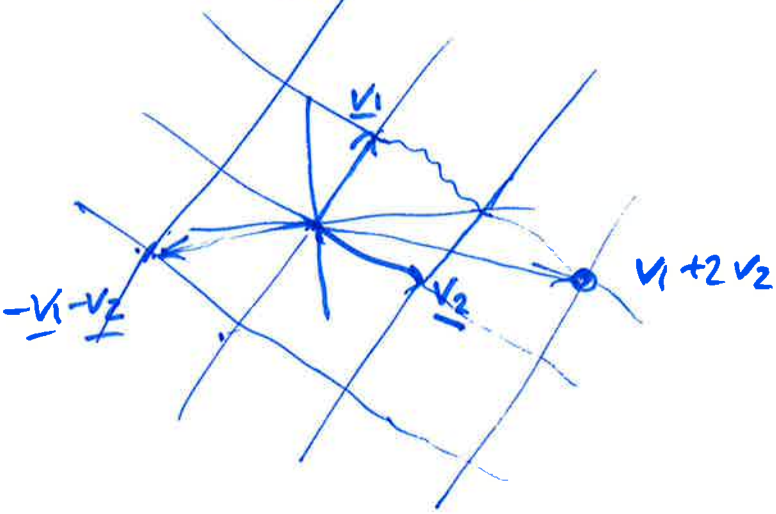
vectors = displacements

$c_1 \cdot \underline{v_1}$ : vector in the same (or opposite) direction as $\underline{v_1}$, stretched with a factor $c_1$.




## Linear combination

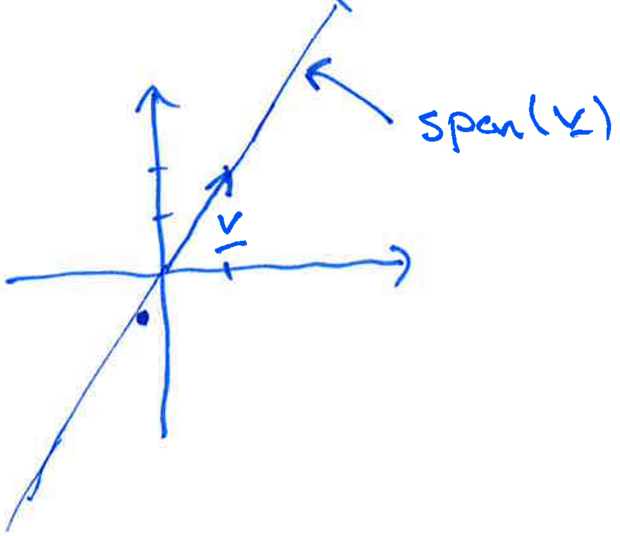
$c_1 \cdot \underline{v_1} + c_2 \cdot \underline{v_2}$



$$\text{span}(\underline{v_1}, \underline{v_2}, \ldots, \underline{v_n}) = \{ c_1\underline{v_1} + c_2\underline{v_2} + \ldots + c_n\underline{v_n} : c_1, c_2, \ldots, c_n \text{ are number} \}$$

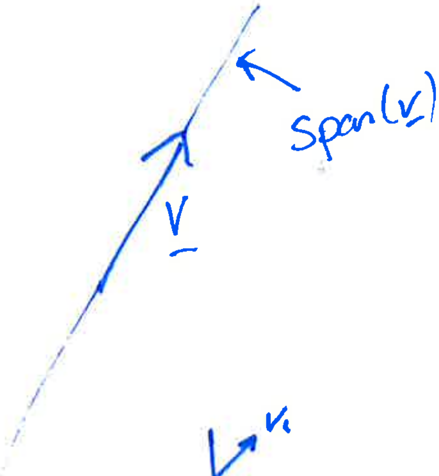
$$= \text{all linear combinations of } \underline{v_1}, \ldots, \underline{v_n}.$$

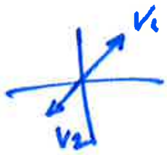
**Ex:** $\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$



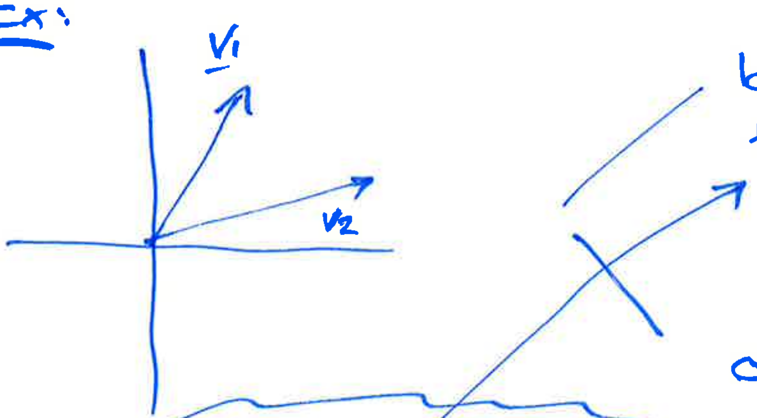

**In general:**

$\underline{v} \neq \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ any non-zero vector




**Ex:**




both vectors lie along the same line: $\text{span}(\underline{v_1}, \underline{v_2})$ is a line

otherwise: $\text{Span}(\underline{v_1}, \underline{v_2})$ is the plane spanned out by these two vectors.

here the dimension of the span drops because

$$\underline{v_2} = c_1 \cdot \underline{v_1} \Rightarrow \text{span}(\underline{v_1}, \underline{v_2}) = \text{span}(\underline{v_1})$$

<u>Ex:</u> Is $\begin{pmatrix}5\\2\end{pmatrix}$ a linear comb. of
$\underline{v}_1 = \begin{pmatrix}1\\2\end{pmatrix}$ and $\underline{v}_2 = \begin{pmatrix}3\\-1\end{pmatrix}$ ?

Is $\begin{pmatrix}5\\2\end{pmatrix}$ in the $\text{span}(\underline{v}_1, \underline{v}_2)$ ?

$$\begin{pmatrix}5\\2\end{pmatrix} = x_1 \cdot \begin{pmatrix}1\\2\end{pmatrix} + x_2 \cdot \begin{pmatrix}3\\-1\end{pmatrix}$$

vector equation

$$\begin{pmatrix}5\\2\end{pmatrix} = \begin{pmatrix}x_1\\2x_1\end{pmatrix} + \begin{pmatrix}3x_2\\-x_2\end{pmatrix}$$

$$\begin{pmatrix}5\\2\end{pmatrix} = \begin{pmatrix}x_1+3x_2\\2x_1-x_2\end{pmatrix} \Longleftrightarrow \begin{aligned} \underline{x_1} + 3x_2 &= 5 \\ 2x_1 - \underline{x_2} &= 2 \end{aligned}$$

<u>linear system</u>

$$|A| = \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} = -1 - 6 = -7 \neq 0$$

$\Downarrow$

one unique solution

$\Downarrow$

Yes, $\begin{pmatrix}5\\2\end{pmatrix}$ is in $\text{span}(\underline{v}_1, \underline{v}_2)$

Ex:

$$\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \underline{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \underline{v}_3 = \begin{pmatrix} 1 \\ 4 \\ 9 \end{pmatrix}$$

Find $\text{span}(\underline{v}_1, \underline{v}_2, \underline{v}_3)$.

$$x_1 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + x_2 \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + x_3 \cdot \begin{pmatrix} 1 \\ 4 \\ 9 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Given $b_1, b_2, b_3$, are there solution to this vector equation?

$$\begin{aligned} x_1 + x_2 + x_3 &= b_1 \\ x_1 + 2x_2 + 4x_3 &= b_2 \\ x_1 + 3x_2 + 9x_3 &= b_3 \end{aligned} \iff \hat{A} = \left(\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 1 & 2 & 4 & b_2 \\ 1 & 3 & 9 & b_3 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 1 & 3 & b_2 - b_1 \\ 0 & 2 & 8 & b_3 - b_1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 1 & 3 & b_2 - b_1 \\ 0 & 0 & 2 & b_3 - b_1 - 2(b_2 - b_1) \end{array}\right)$$

one unique solution for any $b_1, b_2, b_3$ $\Big\}$ consistent

$\Downarrow$

$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is in $\text{span}(\underline{v}_1, \underline{v}_2, \underline{v}_3)$ for all $b_1, b_2, b_3$

$\text{span}(\underline{v}_1, \underline{v}_2, \underline{v}_3)$ = all 3-vectors = $\mathbb{R}^3$

Ex: $\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\underline{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ $\underline{v}_3 = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$

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$$x_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 1 & 2 & 4 & b_2 \\ 1 & 0 & -2 & b_3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 1 & 3 & b_2 - b_1 \\ 0 & -1 & -3 & b_3 - b_1 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 1 & 3 & b_2 - b_1 \\ 0 & 0 & 0 & b_2 + b_3 - 2b_1 \end{array}\right)$$

Cases:

i) $b_2 + b_3 = 2b_1$ : Consistent $\Rightarrow \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is in the span

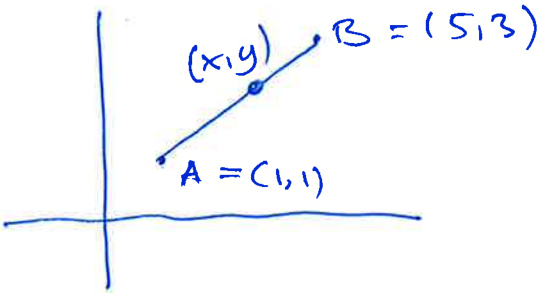
ii) $b_2 + b_3 \neq 2b_1$ : Inconsistent $\Rightarrow \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is not in the span

Conclusion:

$$\text{span}(\underline{v}_1, \underline{v}_2, \underline{v}_3) = \left\{ \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} : b_2 + b_3 = 2b_1 \right\}$$

(2-dimensional)

Ex:




$[A,B]$ = line segment from A to B

$\underline{v}_A = \overrightarrow{OA} = \begin{pmatrix}1\\1\end{pmatrix}$

$\underline{v}_B = \overrightarrow{OB} = \begin{pmatrix}5\\3\end{pmatrix}$

$$\begin{pmatrix}x\\y\end{pmatrix} = \underbrace{\begin{pmatrix}1\\1\end{pmatrix}}_{= v_A} + t\cdot\underbrace{\left[\begin{pmatrix}5\\3\end{pmatrix}-\begin{pmatrix}1\\1\end{pmatrix}\right]}_{= \underline{v}_B - \underline{v}_A = \overrightarrow{AB}} = \begin{pmatrix}1\\1\end{pmatrix} + t\cdot\begin{pmatrix}4\\2\end{pmatrix}, \quad t\in[0,1]$$

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} + t\cdot\begin{pmatrix}4\\2\end{pmatrix} = \begin{pmatrix}1+4t\\1+2t\end{pmatrix}$$

$$\boxed{\left.\begin{aligned} x &= 1+4t \\ y &= 1+2t \end{aligned}\right\} t\in[0,1]}$$

parametric description for the line segment

$$\left.\begin{aligned} x-1 &= 4t \\ y-1 &= 2t \end{aligned}\right\}$$

$$x-1 = 2\cdot(y-1)$$

$$x-1 = 2y-2$$

$$\boxed{x-2y=-1}$$

equation for the line

**In general:** the line segment $[A,B]$ has parametric description

$$\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix} = \underline{v}_A + t\cdot(\underline{v}_B - \underline{v}_A), \quad t\in[0,1]$$

# (2) Linear independence of vectors

$\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n$ : m-vectors

<u>Definition:</u>

The vectors $\underline{v}_1, \ldots, \underline{v}_n$ are called <u>linearly independent</u> if none of the vectors are linear combinations of the other vectors, and <u>linearly dependent</u> if at least one of vectors is a linear comb. of the others.

<u>Ex:</u>

$\underline{v}_2 = 2\underline{v}_1 - \underline{v}_3 \implies \underline{v}_1, \underline{v}_2, \underline{v}_3$ are <u>linearly dependent</u>

$\underline{0} = 2\underline{v}_1 - \underline{v}_2 - \underline{v}_3$

A linear dependency relation can always be written

$$c_1 \cdot \underline{v}_1 + c_2 \underline{v}_2 + \ldots + c_n \cdot \underline{v}_n = \underline{0}$$

with at least one $c_i \neq 0$.

$\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n$ lin. <u>independent</u> $\iff$ $x_1 \underline{v}_1 + \ldots + x_n \underline{v}_n = \underline{0}$ has no non-trivial solution $\underline{x} \neq \underline{0}$

$\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n$ lin. <u>dependent</u> $\iff$ $x_1 \underline{v}_1 + x_2 \underline{v}_2 + \ldots + x_n \underline{v}_n = \underline{0}$ has non-trivial solutions

**Method** to determine if $\underline{v}_1, \dots, \underline{v}_n$ are linearly independent or dependent (and what the linear dependency relation is, if it exists).

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Write down the vector equation $\boxed{x_1 \cdot \underline{v}_1 + x_2 \underline{v}_2 + \dots + x_n \underline{v}_n = \underline{0}}$ and solve it.

$$x_1 \underline{v}_1 + x_2 \underline{v}_2 + \dots + x_n \underline{v}_n = \underline{0}$$

$$\Updownarrow$$

Homogeneous linear system

$$\left(\begin{array}{c:c:c:c|c} & & & & 0 \\ \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_n & 0 \\ & & & & \vdots \\ & & & & 0 \end{array}\right)$$

- Unique solution $\underline{x} = \underline{0}$ (trivial soln.)
- Infinitely many solutions

**Ex:** $\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\underline{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ $\underline{v}_3 = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$ $\boxed{x_1 \underline{v}_1 + x_2 \underline{v}_2 + x_3 \underline{v}_3 = \underline{0}}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 4 & 0 \\ 1 & 0 & -2 & 0 \end{array}\right) \begin{matrix} \\ -1 \\ -1 \end{matrix} \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & -1 & -3 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_3$ free, inf. many solutions.

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$$\left.\begin{array}{ll} x_1 + x_2 + x_3 = 0 & x_1 = 3x_3 - x_3 = 2x_3 \\ \underline{x_2} + 3x_3 = 0 & x_2 = -3x_3 \\ x_3 \text{ free} & x_3 = x_3 \end{array}\right\} \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_3 \\ -3x_3 \\ x_3 \end{pmatrix}$$

$$= x_3 \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

**Hence:** $\underline{v}_1, \underline{v}_2, \underline{v}_3$ are linearly dependent

$x_3 = 1$ gives $\underline{x} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

$$\Downarrow$$

$$2\underline{v}_1 - 3\underline{v}_2 + \underline{v}_3 = \underline{0}$$

$$\boxed{\underline{v}_3 = -2\underline{v}_1 + 3\underline{v}_2}$$

$$\text{Span}(\underline{v}_1, \underline{v}_2, \underline{v}_3) = \text{Span}(\underline{v}_1, \underline{v}_2)$$

**Case I:** $m = n$

$$A = \left( \underline{v}_1 \,\middle|\, \underline{v}_2 \,\middle|\, \cdots \,\middle|\, \underline{v}_n \right)$$

square

$|A| = 0$ : infinitely many solutions
linearly dependent

$|A| \neq 0$ : Unique soln. $\underline{x} = \underline{0}$
linearly independence

$A\underline{x} = \underline{0}$

$\underline{x} = A^{-1} \cdot \underline{0} = \underline{0}$

**Ex:** $\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\underline{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ $\underline{v}_3 = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 0 & -2 \end{vmatrix} = 1 \cdot (4-2) + (-2) \cdot (2-1) = 2 - 2 = \underline{0}$$

linearly dependent

# 3 Linear independence and rank

$\underline{v_1}, \underline{v_2}, \ldots, \underline{v_n}$ : $m$-vectors

$$A = \left( \underline{v_1} \,\middle|\, \underline{v_2} \,\middle|\, \ldots \,\middle|\, \underline{v_n} \right) \rightsquigarrow \text{Compute the rank of } A.$$

$m \times n$-matrix

**Fact:**

The rank of $A$ is equal to the maximal number of linearly independent vectors among $\underline{v_1}, \ldots, \underline{v_n}$.

**Ex:**

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \underline{v_2} = \begin{pmatrix} 7 \\ 4 \\ 11 \end{pmatrix} \quad \underline{v_3} = \begin{pmatrix} 0 \\ -2 \\ -2 \end{pmatrix} \quad v_4 = \begin{pmatrix} 1 \\ 7 \\ 8 \end{pmatrix}$$

$$A = \underbrace{\begin{pmatrix} 1 & 7 & 0 & 1 \\ 2 & 4 & -2 & 7 \\ 3 & 11 & -2 & 8 \end{pmatrix}}_{3\times 4} \begin{matrix} \\ \cdot(-2) \\ \cdot(-3) \end{matrix} \to \begin{pmatrix} 1 & 7 & 0 & 1 \\ 0 & -10 & -2 & 5 \\ 0 & -10 & -2 & 5 \end{pmatrix} \begin{matrix} \\ \\ \cdot(-1) \end{matrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 7 & 0 & 1 \\ 0 & -10 & -2 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$Rk(A) = 2$ : Max number of lin. independent vectors $= 2$

$\underline{\{v_1, v_2\} \text{ are linearly independent}}$ (pivot col.s)

Proof:

$\{\underline{v}_1, \dots, \underline{v}_n\}$ vectors $\longrightarrow$ $\boxed{\begin{array}{l} A = (\underline{v}_1 | \underline{v}_2 | \dots | \underline{v}_n) \\ \text{Lin.sys:} \quad A \cdot \underline{x} = \underline{0} \end{array}}$

rk $A = n$: $n - \text{rk} A = n - n = \underline{0}$ degrees of freedom $\Rightarrow$ unique sol'n $\underline{x} = \underline{0}$ linearly independent

rk $A < n$: $n - \text{rk} A > 0$ free variables $\Rightarrow$ infinitely many sol'ns, linearly dependent

If we take away some vectors from $\underline{v}_1, \dots, \underline{v}_n$, it corresponds to deleting the corresponding col's in $A$. If we remove all non-pivot col's, then there will be no free var's left, and the new system becomes one with unique sol'n $\underline{x} = \underline{0}$, i.e. the remaining vectors ($\Leftrightarrow$ pivot col's) are linearly independent.

## Useful consequences

i) $\dim \operatorname{span}(\underline{v_1}, \underline{v_2}, \dots, \underline{v_n}) = \operatorname{rk} A = \operatorname{rk}\left(\underline{v_1} | \underline{v_2} | \dots | \underline{v_n}\right)$

ii) If $m = n$, then $\boxed{|A| \neq 0 \iff \operatorname{rk} A = n}$

**Ex:** What is the rank of $\begin{pmatrix} 1 & 3 & 2 \\ 2 & 5 & t \\ 4 & 7-t & -6 \end{pmatrix}$ ?

$$\begin{vmatrix} 1 & 3 & 2 \\ 2 & 5 & t \\ 4 & 7-t & -6 \end{vmatrix} = 1\cdot\left(5\cdot(-6) - t(7-t)\right) - 3\left(2(-6) - 4t\right) + 2\cdot\left(2(7-t) - 5\cdot 4\right)$$

$$= -30 - 7t + t^2 + 36 + 12t + 28 - 4t - 40$$

$$= t^2 + t - 6$$

$|A| = 0 \qquad t^2 + t - 6 = 0$

$$t = \frac{-1 \pm \sqrt{1 - 4(-6)}}{2} = -\frac{1}{2} \pm \frac{5}{2} = 2, -3$$

$t \neq -3, 2$: $\quad |A| \neq 0 \Rightarrow \operatorname{rk} A = 3$

$t = -3, 2$: $\quad |A| = 0 \Rightarrow \operatorname{rk} A < 3$

$t = -3$: $\quad A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 5 & -3 \\ 4 & 10 & -6 \end{pmatrix} \qquad \begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} = 5 - 6 = -1 \neq 0$

$\Rightarrow \operatorname{rk} A = 2$

$t = 2$: $\quad A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 5 & 2 \\ 4 & 5 & -6 \end{pmatrix} \qquad \begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} = -1 \neq 0$

$\Rightarrow \operatorname{rk} A = 2$

**Concl:** $\operatorname{rk} A = \begin{cases} 3, & t \neq -3, 2 \\ 2, & t = -3, 2 \end{cases}$