LECTURE 2

GRA 6035 MATHEMATICS

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Plan:

1. Matrices and matrix algebra
2. Determinants
3. Minors, rank and linear algebra

Reading:

[ME] 8.1–8.4, (8.5–8.6), 9.1–9.2, (9.3), 26.1–26.3, (26.4), 26.5

1 Matrices

An $m \times n$-matrix A is a rectangular array of numbers with m rows (#rows $= m$) and n columns (#cols $= n$):

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij})$$

the number in the matrix A in pos. $(1,2)$

row, col.

Ex: $A = \begin{pmatrix} 2 & 3 & 1 \\ -1 & 7 & 4 \end{pmatrix}$

2×3-matrix

Matrix operations

i) Addition / subtraction $\quad A+B,\ A-B$

- defined when A, B have the same size
- computed pos. by pos.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 6 & 2 \end{pmatrix}$$

Zero matrix:

$$O_{2\times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O$$

ii) Multiplication $\quad A \cdot B$

- defined when # cols in A = # rows in B

Ex:

$$\underset{2\times 2}{\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}} \cdot \underset{2\times 3}{\begin{pmatrix} 3 & 2 & 7 \\ 1 & 4 & 1 \end{pmatrix}} = \underset{2\times 3}{\begin{pmatrix} 2 & -2 & 6 \\ 6 & 4 & 14 \end{pmatrix}}$$

$$1\cdot 3 + (-1)\cdot 1 = 2$$

Not symmetric: $\quad A\cdot B \neq B\cdot A$ (in general)

Ex: $\begin{pmatrix} 3 & 2 & 7 \\ 1 & 4 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$ is not defined

$$2\times 3 \quad \neq \quad 2\times 2$$

Identity matrix:

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} = I$$

(n×n identity matrix)

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

It has the property that:

$$\boxed{\begin{aligned} A \cdot I &= A \\ I \cdot B &= B \end{aligned}}$$

Ex:

$$\begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix}$$

iii) Scalar multiplication

$$c \cdot A$$

c: number (scalar)

A: matrix

Ex:

$$2 \cdot \begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 4 \\ 8 & -2 \end{pmatrix}$$

in the example

$$c \cdot A = c \cdot I \cdot A = (cI) \cdot A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 4 \\ 8 & -2 \end{pmatrix}$$

Consequence:

$$c \cdot A = A \cdot c$$

$$(cI) \cdot A = A \cdot (cI)$$

Algebraic laws for matrices

Ex: $$A\cdot(B+C) = AB + AC$$
$$A+B = B+A$$
$$\vdots$$
But: $\boxed{AB \neq BA}$

Ex: $$(A+B)(A-B) = A\cdot A + B\cdot A - A\cdot B - B\cdot B$$
$$= \underline{A^2 + BA - AB \div B^2}$$

Ex: $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$$A\cdot B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq B\cdot A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} : \quad A\cdot C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

iv) Transpose: $A^T = A^t$

Ex: $$\underset{2\times 3}{A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}} \longrightarrow \underset{3\times 2}{A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}}$$

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \longrightarrow A^T = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$$

A is called a <u>symmetric matrix</u> if $A^T = A$.
This is the same as to say that $a_{ij} = a_{ji}$ for all i, j.

Ex: $$A = \begin{pmatrix} 2 & 4 & 2 \\ 4 & 3 & 7 \\ 2 & 7 & -1 \end{pmatrix}$$ is <u>symmetric</u>

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Properties of the transpose:

i) $(A^T)^T = A$

ii) $(A \cdot B)^T = B^T \cdot A^T$

Square matrices: A is square if # rows in A = # cols in A

When A is square, then $A^2, A^3, A^4, \ldots$ is defined

$$A^2 = A \cdot A$$
$$A^3 = A \cdot A \cdot A$$
$$\vdots$$

Ex: $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$

$$A^2 = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 4-1 & 2+0 \\ -2+0 & -1+0 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}$$

$$A^3 = A \cdot A^2 = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}$$

it is possible but difficult to compute

A^{100}, A^n

V) Inverse matrices

A matrix A is said to have an <u>inverse matrix</u> if there is a matrix A^{-1} such that

$$A \cdot A^{-1} = I$$
$$A^{-1} \cdot A = I$$

Ex:
$3^{-1} = \frac{1}{3}$
$3 \cdot \frac{1}{3} = 1$

Ex: $A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$ $\quad A^{-1} = \frac{1}{1} \cdot \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$

$$\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Facts: i) If the inverse exists, it is unique
ii) For the inverse to exist, A must be square and $|A| \neq 0$. Then A is called <u>invertible</u>.

The case: $n=2$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad |A| = ad - bc$

$|A| \neq 0$: $\quad A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$|A| = 0$: $\quad A^{-1}$ does not exist

Remember:

A invertible $\Longleftrightarrow$ A is square with $|A| \neq 0$

Ex: Linear systems in matrix form

$$\boxed{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{array}}$$

$m \times n$ linear system

$A = (a_{ij})$ coeff. matrix

$\hat{A} = (A \mid \underline{b})$ aug. matrix

$\underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$

The same linear system can be written in matrix form:

$$\boxed{A \cdot \underline{x} = \underline{b}}$$

A: coeff. matrix

$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ $\underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

Ex: $\begin{array}{l} x + y = 4 \\ x - y = 2 \end{array}$

$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ $\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ $\underline{b} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

$$A \cdot \underline{x} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ x - y \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$A\underline{x} = \underline{b} \Longleftrightarrow \boxed{\begin{array}{l} x + y = 4 \\ x - y = 2 \end{array}}$$

$$A^{-1} \cdot (A\underline{x}) = A^{-1} \cdot \underline{b}$$

$$\underline{x} = A^{-1}\underline{b} = \overbrace{\frac{1}{-2} \cdot \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}}^{A^{-1}} \cdot \overbrace{\begin{pmatrix} 4 \\ 2 \end{pmatrix}}^{\underline{b}} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

② Determinants

$A = (a_{ij}) \rightsquigarrow \det(A) = |A|$

$n \times n$-matrix

determinant of A, it is a number

$n = 2$:

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightsquigarrow |A| = ad - bc$

$n = 3$:

$A = \left(\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}\right) \begin{matrix} a & b \\ d & e \\ g & h \end{matrix}$

Only works for $n=3$

$|A| = aei + bfg + cdh$
$\quad - ceg - afh - bdi$

Ex:

$A = \left(\begin{matrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{matrix}\right) \begin{matrix} 1 & 1 \\ 1 & -1 \\ 1 & 2 \end{matrix}$

$|A| = -4 + 1 + 2$
$\quad - (-1) - 2 - 4 = \underline{-6}$

In general: (works for any n)
- Cofactor expansion
- Gaussian elimination

i) Cofactor expansion

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & & \vdots \end{pmatrix}$$

← along the first row

$n \times n$-matrix

$$|A| = a_{11} \cdot C_{11} + a_{12} \cdot C_{12} + \cdots + a_{1n} \cdot C_{1n}$$

C_{ij} : the cofactor in position (i,j)

$$C_{ij} = \underbrace{(-1)^{i+j}}_{\text{sign}} \cdot \underbrace{M_{ij}}_{\text{minor}}$$

M_{ij} : minor in position (i,j)

M_{ij} = determinant of the submatrix you get when you delete row $\#i$, col $\#j$.

Ex:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix}$$

Signs: $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

$$|A| = 1 \cdot (+1) \cdot M_{11} + 1 \cdot (-1) \cdot M_{12} + 1 \cdot (+1) \cdot M_{13}$$

$$= + \begin{vmatrix} -1 & 1 \\ 2 & 4 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix}$$

$$= -6 - 3 + 3 = \underline{\underline{-6}}$$

Facts:

i) Cofactor expansion works for any $n \times n$-matrix

ii) Cofactor expansion along any row or column gives the same result.

Ex:

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 7 & 5 \\ 0 & 0 & 13 \end{pmatrix}$$

$$|A| = +1 \cdot \begin{vmatrix} 7 & 5 \\ 0 & 13 \end{vmatrix} - 0 \cdot \ldots + 0 \cdot \ldots$$

$$= 1 \cdot 7 \cdot 13 = \underline{\underline{91}}$$

Fact: A is called <u>upper triangular</u> if it is square with only zeros under the diagonal,

$$a_{ij} = 0 \quad \text{when} \quad i > j$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

$$\boxed{\text{If } A \text{ is upper triangular, then } |A| = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn}}$$

ii) Determinants via Gaussian elimination

Ex:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \qquad |A| = |E| = \underline{\underline{-6}}$$

$$\downarrow \; (-1)$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 1 & 3 \end{pmatrix} \xrightarrow{:(-2)} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix} \xrightarrow{-1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$|E| = 3$$

$$\frac{|A|}{-2} = |E| = 3 \Rightarrow |A| = -6$$

$$\downarrow \; \left(\tfrac{1}{2}\right)$$

echelon form, upper triangular

$$E = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \qquad |E| = 1 \cdot (-2) \cdot 3 = \underline{-6}$$

Elementary row operations:

- If you add a multiple of one row to another row, the determinant does not change.
- If you switch two rows, the determinant changes with a factor of $\underline{-1}$.
- If you multiply a row with $c \neq 0$, then the determinant changes with a factor of $\underline{c}$.

Properties of the determinant

i) $|AB| = |A| \cdot |B|$

ii) $|A^T| = |A|$

iii) $|A^{-1}| = \dfrac{1}{|A|}$

iv) $|cA| = c^n \cdot |A|$ when c is a number, A is $n \times n$-matrix

Exercise:

Compute
$$\begin{vmatrix} 4 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 6 & -2 & 0 \\ 1 & -1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{vmatrix}.$$

Sol:

$$\begin{vmatrix} 4 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 6 & -2 & 0 \\ 1 & -1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{vmatrix} \quad -\tfrac{1}{4} \quad -1/4$$

$$= \begin{vmatrix} 4 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 6 & -2 & 0 \\ 0 & -1 & 2 & 1/4 & 1/4 \\ 0 & 1 & 0 & 1/4 & 1/4 \end{vmatrix}$$

$$= 4 \begin{vmatrix} 2 & 0 & 1 & -1 \\ 0 & 6 & -2 & 0 \\ -1 & 2 & 1/4 & 1/4 \\ 1 & 0 & 1/4 & 1/4 \end{vmatrix} \quad 1 \quad -2$$

$$= 4 \cdot \begin{vmatrix} 0 & 0 & 1/2 & -3/2 \\ 0 & 6 & -2 & 0 \\ 0 & 2 & 1/2 & 1/2 \\ 1 & 0 & 1/4 & 1/4 \end{vmatrix}$$

$$= 4 \cdot 1 \cdot (-1) \cdot \begin{vmatrix} 0 & 1/2 & -3/2 \\ 6 & -2 & 0 \\ 2 & 1/2 & 1/2 \end{vmatrix}$$

$$= -4 \cdot \left(-6 \cdot \left(1/4 + 3/4\right) + 2(-3)\right)$$

$$= -4 \cdot (-6 - 6) = -4 \cdot (-12) = \underline{\underline{48}}$$

3 Minors and rank

A $m \times n$ matrix

A <u>minor</u> of order r in A is the determinant of an $r \times r$ Submatrix of A.

<u>Ex:</u> $A = \begin{pmatrix} 1 & 2 & 4 \\ 7 & -1 & 3 \end{pmatrix}$ 2×3

<u>Order 2 minors:</u> Delete one column

$M_{12,12} = \begin{vmatrix} 1 & 2 \\ 7 & -1 \end{vmatrix} = \underline{-15}$

$M_{12,13} = \begin{vmatrix} 1 & 4 \\ 7 & 3 \end{vmatrix} = \underline{-25}$

$M_{12,23} = \begin{vmatrix} 2 & 4 \\ -1 & 3 \end{vmatrix} = \underline{10}$

Since we have a 2-minor that is non-zero, the rank of A is 2.
(2-minor = max. minor)

<u>Order 1 minors:</u>

$M_{1,1} = \underline{1}$ $M_{1,2} = \underline{2}$ $M_{1,3} = \underline{4}$

$M_{2,1} = \underline{7}$ $M_{2,2} = \underline{-1}$ $M_{2,3} = \underline{3}$

<u>Fact:</u>

The rank of a matrix is equal to the <u>maximal</u> <u>order</u> of a non-zero minor.

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Ex: Minors and linear systems

$$x + y + z + w = 4$$
$$x + 2y + 3z - w = 2$$
$$x + 3y + z = 1$$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & -1 \\ 1 & 3 & 1 & 0 \end{pmatrix}$$

Maximal minors

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 1 \end{vmatrix} = 1 \cdot (-7) - 1 \cdot (-2) + 1 \cdot 1 = -4 \neq 0$$

$\Downarrow$

- rank is 3
- the pivot positions are inside this submatrix
- w is a free variable
  inf. many solutions

$$\hat{A} = \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 1 & 2 & 3 & -1 & 2 \\ 1 & 3 & 1 & 0 & 1 \end{array}\right)$$

w free

Solution:

w is free, move to the other side

$$x + y + z = 4 - w$$
$$x + 2y + 3z = 2 + w$$
$$x + 3y + z = 1$$

matrix corresponds to non-zero minor, so its determinant is non-zero and the inverse exists

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4-w \\ 2+w \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 4-w \\ 2+w \\ 1 \end{pmatrix}$$

When we solve a linear system using minors:

Assume that $\text{rk}\, A = \text{rk}\, \hat{A} = r$, and choose a non-zero minor in A of order r.

* Look at the r equations "running through" the minor chosen, and ignore the rest.
* Solve these equations for the variables "in the minor", the rest are free variables.

$$\left.\begin{aligned} x+y+z+w &= 4 \\ x-2y+3z-w &= 2 \\ x+3y-z &= 1 \end{aligned}\right\} \qquad A = \begin{pmatrix} \boxed{\begin{matrix} 1 & 1 & 1 \\ 1 & -2 & 3 \\ 1 & 3 & -1 \end{matrix}} & \begin{matrix} 1 \\ -1 \\ 0 \end{matrix} \end{pmatrix}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -2 & 3 \\ 1 & 3 & -1 \end{vmatrix} = -4 \neq 0$$

maximal non-zero minor

$$\begin{aligned} x+y+z &= 4-w \\ x-2y+3z &= 2+w \\ x+3y-z &= 1 \end{aligned}$$

Use eqn. 1-3

Solve for x, y, z; w is free

Solve for x, y, z

$\Downarrow$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 3 \\ 1 & 3 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 4-w \\ 2+w \\ 1 \end{pmatrix}$$