LECTURE 14

GRA 6035 MATHEMATICS

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Plan:

* Review (continued): Optimization, Differential equations
* Final Exam 12/2014: Question 3–4 (5)

A) Differential equations / Difference equations.

Differential equations:

First order: $y' = F(y,t)$

a) Linear: $y' + \underline{a(t)} \cdot y = f(t)$

Methods: 1) Integrating factor

$$u = e^{\int a(t)\,dt}$$

Multiplying with int. factor:

$$(u(t)\cdot y)' = f(t)\cdot u(t)$$

$$\boxed{y = \frac{1}{u(t)} \int f(t)\cdot u(t)\,dt}$$

Works if $a(t) = a$ is a constant. →

2) Superposition principle

$$y = y_h + y_p$$

$$= C\cdot e^{-at} + y_p$$

$y' + ay = 0$

$r + a = 0$

$r = -a$

Second order:

Linear: $y'' + ay' + by = f(t)$

Method: Superposition principle

$$y = y_h + y_p$$

y_h: $y'' + ay' + by = 0$

$r^2 + ar + b = 0$

Find y_h from the roots:

$r_1 \neq r_2$: $y_h = C_1 \cdot e^{r_1 t} + C_2 \cdot e^{r_2 t}$

$r_1 = r_2$: $y_h = C_1 e^{r_1 t} + C_2 t e^{r_2 t}$

How to find y_p: $\quad y'' + ay' + by = f(t)$

Start with $f(t)$, compute f' (and maybe f'')
Make a guess $y(t)$ based on the form of f, f', f'' with undetermined coeffs.
Substitute y into the diff.-eqn. and try to adjust the coeffs. to find a solution.

If it doesn't work, try to multiply with t.

First order:

b) Separable: $y' = f(y) \cdot g(t)$

Method: $\int \frac{1}{f(y)} y' dt = \int g(t)\, dt$

$$\boxed{\int \frac{1}{f(y)} dy = \int g(t)\, dt}$$

c) Exact: $p(y,t) + q(y,t) \cdot y' = 0$ is exact if there is an $h = h(y,t)$ s.t.

Method:
$$\begin{cases} \frac{\partial h}{\partial t} + \frac{\partial h}{\partial y} \cdot \frac{dy}{dt} = 0 \\ \text{In that case: } \boxed{h(y,t) = C} \end{cases}$$

Difference equations

Linear first order: $y_{t+1} + a y_t = f_t$

Superposition principle:

$y_t = y_t^h + y_t^p$

$= C \cdot (-a)^t + y_t^p$

Linear second order:

$y_{t+2} + a y_{t+1} + b y_t = f_t$

Method: $y_t = y_t^h + y_t^p$

Char. eqn: $r^2 + ar + b = 0$

$r_1 \neq r_2$: $y_t^h = C_1 r_1^t + C_2 r_2^t$

$r_1 = r_2$: $y_t^h = C_1 r_1^t + C_2 t r_2^t$

Final exam 12/2014

Question 3

a) $y_{t+1} - 3y_t = -5(2t+1)$ ← Difference eqn. Linear first order

$$y_t = y_t^h + y_t^p = \underline{\underline{C \cdot 3^t + 5t + 5}}$$

$\underline{y_t^h}$: $y_{t+1} - 3y_t = 0$

$r - 3 = 0 \quad \underline{r = 3} \quad \Rightarrow y_t^h = \underline{C \cdot 3^t}$

$\underline{y_t^p}$: $y_{t+1} - 3y_t = -5(2t+1) = -10t - 5$

$f_t = -10t - 5$

$f_{t+1} = -10(t+1) - 5 = -10t - 15$

Guess: $y_t = \underline{At + B}$

$y_{t+1} = A \cdot (t+1) + B$

$= \underline{At + A + B}$

$$(At + A + B) - 3(At + B) = -10t - 5$$

$$\underline{(-2A)}t + (\underline{A - 2B}) = \underline{-10}t \underline{-5}$$

$-2A = -10 \qquad \underline{A = 5}$

$A - 2B = -5 \qquad 5 - 2B = -5$

$-2B = -10$

$B = \underline{5}$

$y_t^p = At + B$

$= \underline{5t + 5}$

b) $t^3 y' = y^2$

first order differential eqn.
not linear

separable

$y' = \frac{1}{t^3} y^2$

$\frac{1}{y^2} y' = \frac{1}{t^3}$

$\int \frac{1}{y^2} y' \, dt = \int \frac{1}{t^3} dt$

$\int y^{-2} dy = \int t^{-3} dt$

$\frac{y^{-1}}{-1} = \frac{t^{-2}}{-2} + C$

$-\frac{1}{y} = -\frac{1}{2} \frac{1}{t^2} + C$

$1/y = \frac{1}{2t^2} - C$

$y = \frac{1}{\frac{1}{2t^2} - C} \cdot \frac{2t^2}{2t^2} = \frac{2t^2}{1 - 2ct^2}$

c) $(2yt-1) y' = (t+1)e^t - y^2$

Exact?

$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial y} \cdot y' = 0$

$(y^2 - (t+1)e^t) + (2yt-1) y' = 0$

$$\boxed{\begin{aligned} h'_t &= y^2 - (t+1)e^t \\ h'_y &= 2yt - 1 \end{aligned}}$$

$\boxed{\text{Solution: } h(y,t) = C}$

$h = y^2 t - y + C(t)$

$h'_t = \cancel{y^2} + C'(t) = \cancel{y^2} - (t+1)e^t$

$$C'(t) = -(t+1)e^t$$

$$\boxed{\begin{array}{ll} u = -(t+1) & v = e^t \\ u' = -1 & v' = e^t \end{array}}$$

$$C(t) = \int -(t+1)e^t\,dt$$

$$= \int uv'\,dt = uv - \int u'v\,dt$$

$$= -(t+1)e^t - \int (-1)\cdot e^t\,dt$$

$$= -(t+1)e^t + e^t + C$$

$$= (-t \cancel{-1} \cancel{+1})e^t + C = -te^t + C$$

$$h = \underline{y^2 t - y - te^t}$$

Solution: $y^2 t - y - te^t = K$

$$t y^2 - 1 y + (-te^t - K) = 0$$

$$y = \frac{1 \pm \sqrt{1 - 4t(-te^t - K)}}{2t}$$

$$= \underline{\underline{\frac{1 \pm \sqrt{1 + 4t(te^t + K)}}{2t}}}$$

B) Constrained optimization

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Lagrange problem: Optimization, Equality constraints

max / min $f(x_1, \ldots, x_n)$ when $\begin{cases} g_1(x_1, \ldots, x_n) = a_1 \\ g_2(x_1, \ldots, x_n) = a_2 \\ \vdots \\ g_m(x_1, \ldots, x_n) = a_m \end{cases}$

$$\mathcal{L} = f(\underline{x}) - \lambda_1 \cdot g_1(\underline{x}) - \lambda_2 \cdot g_2(\underline{x}) - \ldots - \lambda_m g_m(\underline{x})$$

FOC: $\mathcal{L}'_{x_1} = 0$, $\mathcal{L}'_{x_2} = 0$, $\vdots$, $\mathcal{L}'_{x_n} = 0$

C: $g_1(\underline{x}) = a_1$, $\vdots$, $g_m(\underline{x}) = a_m$

FOC + C = Lagrange conditions

Solutions of Lagrange conditions $\Rightarrow$ Candidates for max / min.

Kuhn-Tucker problem: Optimization, closed inequality constraints ($\leq$)

max $f(x_1, \ldots, x_n)$ when $\begin{cases} g_1(\underline{x}) \leq a_1 \\ g_2(\underline{x}) \leq a_2 \\ \vdots \\ g_m(\underline{x}) \leq a_m \end{cases}$ $\leftarrow$ Standard form: max $\leq a_i$

$$L = f(\underline{x}) - \lambda_1 g_1(\underline{x}) - \ldots - \lambda_m g_m(\underline{x})$$

FOC: $\mathcal{L}'_{x_1} = 0$, $\mathcal{L}'_{x_2} = 0$, $\vdots$, $\mathcal{L}'_{x_n} = 0$

C: $g_1(\underline{x}) \leq a_1$, $\vdots$, $g_m(\underline{x}) \leq a_m$

CSC:

FOC + C + CSC = Kuhn-Tucker conditions

FOC: $\mathcal{L}'_{x_1} = 0$
$\vdots$
$\mathcal{L}'_{x_n} = 0$

C: $g_1(\underline{x}) \leq a_1$
$g_2(\underline{x}) \leq a_2$
$\vdots$
$g_m(\underline{x}) \leq a_m$

CSC:

$\lambda_1 \geq 0$ and $\lambda_1 \cdot (g_1(\underline{x}) - a_1) = 0$

$\lambda_2 \geq 0$ " $\lambda_2 (g_2(\underline{x}) - a_2) = 0$

$\vdots$

$\lambda_m \geq 0$ " $\lambda_m (g_m(\underline{x}) - a_m) = 0$

Solutions of Kuhn-Tucker conditions $\Rightarrow$ candidates for max

Note:

$g_i(\underline{x}) \leq a_i$, $\lambda_i \geq 0$ and $\lambda_i (g_i(\underline{x}) - a_i) = 0$

can be written:

(binding) $g_i(\underline{x}) = a_i$	(non-binding) $g_i(\underline{x}) < a_i$
$\lambda_i \geq 0$	$\lambda_i = 0$

Methods to conclude

i) Second order conditions (SOC)

If $(\underline{x}^*; \lambda^*)$ is a candidate point that satisfy Lagrange / KT-conditions, then

* $H(\mathcal{L}(\underline{x}; \lambda^*))$ neg. semidefn. for all $\underline{x}$ ($\mathcal{L}(\underline{x}; \lambda^*)$ is concave)
  $\Rightarrow$ $\underline{x}^*$ is max
* $H(\mathcal{L}(\underline{x}; \lambda^*))$ pos. semidefn. for all $\underline{x}$ ($\mathcal{L}(\underline{x}; \underline{\lambda}^*)$ is convex)
  $\Rightarrow$ $\underline{x}^*$ is min

ii) Extreme Value thm + Exhaustion.

Extreme Value thm (EVT):

If the set of adm. pts (points that satisfy all constraints) is bounded, then there is a max and a min.

Exhaustion: Check all possibilities

i) All regular candidates, i.e. solutions to the Lagrange / KT-conditions.

ii) All adm. points that fails NDCQ.

NDCQ: $g(\underline{x}) = a$	$g(\underline{x}) < a$
$rk\,(g'_{x_1}\; g'_{x_2} \cdots g'_{x_n}) = 1$	no condition

Questin 4.

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max $f = x + 4y + 2z + 5w$ when $2x^2 + 2y^2 + 2z^2 + 2w^2 + 2yz \le 21$

Std form.

a) $L = x + 4y + 2z + 5w - \lambda \cdot (2x^2 + 2y^2 + 2z^2 + 2w^2 + 2yz)$

FOC:

$L'_x = 1 - \lambda \cdot 4x = 0$

$L'_y = 4 - \lambda \cdot (4y + 2z) = 0$

$L'_z = 2 - \lambda \cdot (2y + 4z) = 0$

$L'_w = 5 - \lambda \cdot 4w = 0$

C + CSC:

a) $2x^2 + 2y^2 + 2z^2 + 2w^2 + 2yz = 21$
$\lambda \geq 0$

or

b) $\cancel{2x^2 + 2y^2 + 2z^2 + 2w^2 + 2yz < 21}$
$\cancel{\lambda = 0}$

Case b): $\lambda = 0$ impossible (because of FOC's)

$\Rightarrow$ no candidates

Case a):

FOC: $1 - \lambda \cdot 4x = 0$

$4\lambda x = 1$

$x = 1/4\lambda$

$5 = 4\lambda \cdot w$

$w = 5/4\lambda$

$4 = \lambda \cdot (4y + 2z)$

$2 = \lambda \cdot (2y + 4z)$ $\cdot(-2)$

$0 = \lambda \cdot (0 - 6z)$

$0 = -6\lambda z$

$\cancel{\lambda = 0}$ or $z = 0$

$4 = \lambda \cdot (4y)$

$y = \frac{4}{4\lambda}$

FOC:

$x = \frac{1}{4\lambda} \quad y = \frac{4}{4\lambda} \quad z = 0 \quad w = \frac{5}{4\lambda}$

$2x^2 + 2y^2 + 2yz + 2z^2 + 2w^2 = 21 \quad , \quad \lambda \geq 0$

$2 \cdot \left(\frac{1}{4\lambda}\right)^2 + 2 \cdot \left(\frac{4}{4\lambda}\right)^2 + 2 \cdot \left(\frac{5}{4\lambda}\right)^2 = 21$

$$\frac{2 + 32 + 50}{(4\lambda)^2} = 21$$

$\frac{84}{(4\lambda)^2} = 21 \qquad \frac{84}{21} = (4\lambda)^2$

$(4\lambda)^2 = 4$

$4\lambda = \pm 2$

$\lambda = \pm 2/4 = \pm 1/2$

$x = 1/2, \quad y = 2, \quad z = 0, \quad w = 5/2, \quad \underline{\lambda = 1/2}$

One regular candidate pt: $(x, y, z, w; \lambda) = \underline{(1/2, 2, 0, 5/2; 1/2)}$

b) $f(1/2, 2, 0, 5/2) = \frac{1}{2} + 4 \cdot 2 + 2 \cdot 0 + 5 \cdot 5/2 = 8 + \frac{26}{2}$
$= \underline{\underline{21}}$

SOC: $\mathcal{L}(x, y, z, w; 1/2) = x + 4y + 2z + 5w$
$- \frac{1}{2}(2x^2 + 2y^2 + 2yz + 2z^2 + 2w^2)$

$$H(\mathcal{L}) = \left(\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & -1 & -2 & 0 \\ \hline 0 & 0 & 0 & -2 \end{array}\right)$$

neg. semidefinite?

$$D_1 = -2 < 0$$
$$D_2 = 4 > 0$$
$$D_3 = -2 \cdot (4-1) = -6 < 0$$
$$D_4 = -2 \cdot D_3 = 12 > 0$$

$\Bigg\}$ neg. defn.

$\Downarrow$

neg. semidefn.

$\Downarrow$

$h(x,y,z,w;\ 1/2)$ concave

$\Downarrow$

$(1/2, 2, 0, 5/2)$ is max by the SOC, with max value $\underline{\underline{21}}$

c) max $x + \underline{\underline{3.8}}y + 2z + \underline{\underline{5.4}}w$ when ---- (as before)

KT-problem with parameters:

max $x + ay + 2z + bw$ when $2x^2 + 2y^2 + 2yz + 2z^2 + 2w^2 \leq 21$

$f^*(a,b)$: max value when the parameters are a, b

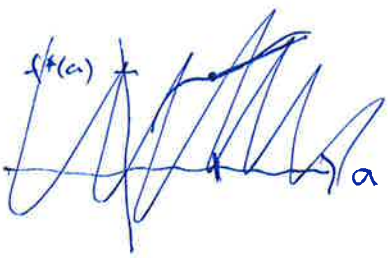
We know: $f^*(4,5) = 21$

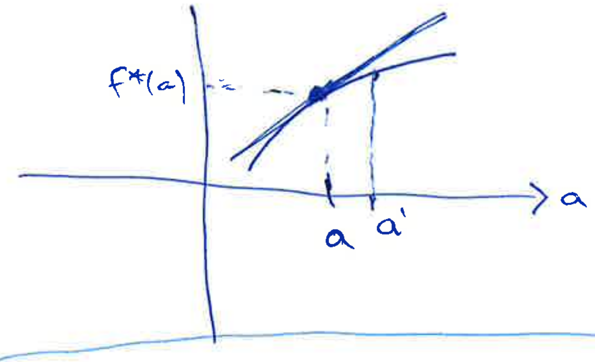
We want to estimate: $f^*(3.8, 5.4) = ?$

Linear approx: $f^*(3.8, 5.4) \simeq f^*(4,5) + \Delta a \cdot \frac{df^*}{da} + \Delta b \cdot \frac{df^*}{db}$

$$= \underline{\underline{21 - 0.2 \cdot \frac{df^*}{da} + 0.4 \cdot \frac{df^*}{db}}}$$

Envelope Thm.!





$$f^*(a') \simeq f^*(a) + \Delta a \cdot \frac{\partial f^*}{\partial a}$$

<u>Envelope Thm.:</u> $$\frac{df^*(a)}{da} = \frac{\partial L}{\partial a}\Bigg|_{\underline{x}=\underline{x}^*(a),\ \underline{\lambda}=\lambda^*(a)}$$

$$L = x + \underline{ay} + 2z + \underline{bw} - \lambda \cdot \left(2x^2 + 2y^2 + 2yz + 2z^2 + 2w^2 - 21\right)$$

$$\left.\begin{aligned} \frac{\partial L}{\partial a} &= y \\ \frac{\partial L}{\partial b} &= w \end{aligned}\right\}$$

<u>Env. Thm.:</u> $$\frac{\partial f^*(a,b)}{\partial a} = y^*(a,b)$$

$$\frac{\partial f^*(a,b)}{\partial b} = w^*(a,b)$$

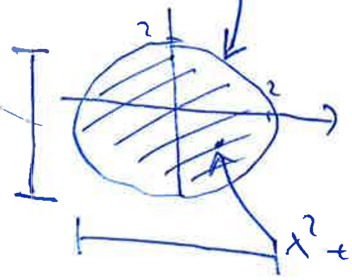
$$\Downarrow$$

$$f^*(3.8,\ 5.4) \simeq f^*(4,5) - 0.2 \cdot \overbrace{2}^{y^*(4,5)} + 0.4 \cdot \overbrace{\frac{5}{2}}^{w^*(4,5)}$$

$$= 21 - 0.4 + 1 = \underline{\underline{21.6}}$$

How to check that a set is bounded?

Ex: $x^2+y^2 \le 4$ $\longrightarrow$ $x^2+y^2=4$



$x^2+y^2<4$

it is **bounded**.

Question: is there a smallest and largest value of x among adm. pts.?

is there a smallest and largest value of y among adm pts?