LECTURE 13

GRA 6035 MATHEMATICS BI

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Plan:

1. Review
2. Final exam 12/2014

Same exam structure as before:

12 problems : 6p each = 72p : (72p = 100%)
1 extra problem : 6p ... 6p

78p

Grading scale last year:

A: 92% B: 77% C: 58% D: 46% E: 40% (of 72p)

a) Matrix methods
b) Unconstrained optimization
c) Constrained optimization
d) Differential / difference equations

} Next time? (Dec 08th)

1 Matrix methods

Basic techniques:
- a) Solving linear systems (Gaussian elimination)
- b) Determinants

i) Linear independence of vectors

$\{\underline{v_1}, \underline{v_2}, \ldots, \underline{v_n}\}$: m-vectors $\rightsquigarrow$ $A = (\underline{v_1} | \underline{v_2} | \ldots \underline{v_n})$

Are the vectors linearly independent?

Fact: If $m=n$, then

$|A| \neq 0 \iff$ linearly independent
$|A| = 0 \iff$ linearly dependent

Fact: $A\underline{x} = \underline{0}$ ↑ Solve

only the trivial solution $\iff$ linearly independent

$\underline{x} = \underline{0}$ ⇕ no free variables

Ex: $A = \begin{pmatrix} 1 & t & -2 \\ 2 & 4 & -t \\ -t & -4 & -4 \end{pmatrix}$ $\quad \underline{v_1} = \begin{pmatrix} 1 \\ 2 \\ -t \end{pmatrix}$ $\underline{v_2} = \begin{pmatrix} t \\ 4 \\ -4 \end{pmatrix}$ $\underline{v_3} = \begin{pmatrix} -2 \\ -t \\ -4 \end{pmatrix}$

$$\begin{vmatrix} 1 & t & -2 \\ 2 & 4 & -t \\ -t & -4 & -4 \end{vmatrix} = 1 \cdot (-16-4t) - 2(-4t-8) - t(-t^2+8)$$

$$= \cancel{-16} - 4t + \cancel{8t} + \cancel{16} + \underline{t^3} - \cancel{8t}$$

$$= t^3 - 4t = t(t^2-4) = t(t-2)(t+2)$$

$t = 0, 2, -2$: $|A| = 0 \iff$ vector ~~are~~ lin. dependent

$t \neq 0, 2, -2$: $|A| \neq 0 \iff$ vectors are lin. independent

ii) Rank

A $m \times n$-matrix: $rk(A)$ = max number of linearly independent column vectors in A

Fact I:

If $m=n$, then $\begin{cases} rk\,A = n \iff |A| \neq 0 \\ rk\,A < n \iff |A| = 0 \end{cases}$

Fact 2:

$rk(A) = \#$ pivot positions in A

Ex: $A = \begin{pmatrix} 1 & t & -2 \\ 2 & 4 & -t \\ -t & -4 & -4 \end{pmatrix}$

$|A| = t^3 - 4t \Rightarrow \begin{cases} |A| = 0 \iff t = 0, 2, -2 \to rk\,A < 3 \\ |A| \neq 0 \iff t \neq 0, 2, -2 \to rk\,A = 3 \end{cases}$

If $t=-2$: $A = \begin{pmatrix} \boxed{\begin{matrix} 1 & -2 \\ 2 & 4 \end{matrix}} & \begin{matrix} -2 \\ 2 \end{matrix} \\ \begin{matrix} 2 & -4 \end{matrix} & -4 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -2 \\ 0 & 8 & 6 \\ 0 & 0 & 0 \end{pmatrix}$ rk A = 2 when $t=-2$

$\begin{vmatrix} 1 & -2 \\ 2 & 4 \end{vmatrix} = 4 + 4 = 8 \neq 0$ rk A = 2 when $t=-2$

2-minor

$\boxed{rk\,A = \text{maximal order of a non-zero minor of } A}$

iii) Eigenvalues and eigenvectors

A $n\times n$-matrix: If $A\underline{x} = \lambda \underline{x}$ with $\underline{x} \neq \underline{0}$ then

$$A\underline{x} = \lambda \underline{x} \quad \Updownarrow \quad (A-\lambda I)\underline{x} = 0$$

the number λ is a eigenvalue

the vector $\underline{x}$ is a eigenvector

Fact 1: The eigenvalues are the solutions of the characteristic equation

$$\boxed{\det(A - \lambda I) = 0} \qquad I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Fact 2: The eigenvectors for A with eigenvalue λ^* are the solutions of

$$\boxed{(A - \lambda^* I)\underline{x} = \underline{0}}$$

Fact 3: There is a diagonal matrix D and an invertible matrix P such that

$$\boxed{P^{-1}AP = D} \qquad \text{(diagonalization)}$$

if and only if i) there are n eigenvalues (when you count with multiplicities) $\lambda_1, \lambda_2, \ldots, \lambda_n$

ii) there are n linearly independent eigenvectors $\underline{v_1}, \ldots, \underline{v_n}$

If this is the case,

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{pmatrix} \qquad P = \left(\underline{v_1} \,\vdots\, \underline{v_2} \,\vdots\, \cdots \,\vdots\, \underline{v_n} \right)$$

Ex: $A = \begin{pmatrix} 1 & -2 & -2 \\ 2 & 4 & 2 \\ 2 & -4 & -4 \end{pmatrix}$

Eigenvalues: $|A - \lambda I| = \begin{vmatrix} 1-\lambda & -2 & -2 \\ 2 & 4-\lambda & 2 \\ 2 & -4 & -4-\lambda \end{vmatrix} = 0$

$$(1-\lambda)\cdot\Big(\big((4-\lambda)(-4-\lambda)+8\big) - 2\cdot\big(\cancel{8}+2\lambda-\cancel{8}\big) + 2\cdot\big(-4+8-2\lambda\big)$$

$+8(1-\lambda)$

$-8(\lambda-1)$

$$= (1-\lambda)(4-\lambda)(-4-\lambda) + 8(1-\lambda) - 8\lambda + 8$$

$$= (1-\lambda)\cdot\big[(4-\lambda)(-4-\lambda)+8+8\big]$$

$$= (1-\lambda)\cdot(\lambda^2) = \lambda^2(1-\lambda) = 0$$

$\lambda^2 - \lambda^3$

$= \lambda^2(1-\lambda)$

$\underline{\lambda_1 = 0} \quad \underline{\lambda_2 = 0} \quad \underline{\lambda_1 = 1} \rightsquigarrow D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Eigenvectors

$P = \left(\underline{v_1} \mid \underline{v_2} \mid \underline{v_3}\right)$

$\begin{pmatrix} 1/2 \\ -3/4 \\ 1 \end{pmatrix}$?

$\underline{\lambda = 0}:$ $\begin{array}{l} \\ -2 \\ -2 \end{array}\begin{pmatrix} 1 & -2 & -2 \\ 2 & 4 & 2 \\ 2 & -4 & -4 \end{pmatrix} \underline{x} = \underline{0}$

$\downarrow$

$\begin{pmatrix} 1 & -2 & -2 \\ 0 & 8 & 6 \\ 0 & 0 & 0 \end{pmatrix} \underline{x} = \underline{0}$

one free variable

$\underline{x} - 2y - 2z = 0$

$8\underline{y} + 6z = 0$

z free

$y = -\frac{6}{8}z = -\frac{3}{4}z$

$x = 2y + 2z$

$= 2z - \frac{3}{2}z$

$= \frac{1}{2}z$

Eigenvect. for $\lambda = 0$:

$$\underline{x} = \begin{vmatrix} x \\ y \\ z \end{vmatrix} = \begin{pmatrix} 1/2\, z \\ -3/4\, z \\ z \end{pmatrix} = z\cdot\begin{pmatrix} 1/2 \\ -3/4 \\ 1 \end{pmatrix}$$

A is not diagonalizable.

If λ has multiplicity m, then $(A-\lambda I)\underline{x}=\underline{0}$ must have m free variables for A to be diagonalizable.

If A is symmetric, then it is diagonalizable

Fact: If A (nxn-matrix) has eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, then

$$\lambda_1+\lambda_2+\dots+\lambda_n = tr(A) = a_{11}+a_{22}+\dots+a_{nn}$$
$$\lambda_1 \cdot \lambda_2 \cdot \dots \cdot \lambda_n = det(A)$$

V) Definiteness: What is the definiteness of A

A symmetric nxn-matrix $\Big\}$ Leading principal minors: $D_1, D_2, \dots, D_n$
Principal minors: $\Delta_1, \Delta_2, \dots, \Delta_n$ (several of each order)

Fact:

i) $D_1>0, D_2>0, \dots, D_n>0$: pos. definite
ii) $D_1<0, D_2>0, D_3<0, \dots$: neg. definite
iii) $D_1 \geq 0, D_2 \geq 0, \dots, D_n \geq 0$: pos. semidefinite or indefinite
iv) $D_1 \leq 0, D_2 \geq 0, D_3 \leq 0, \dots$: neg. semidefinite or indefinite
v) All other cases: indefinite

$\Delta_1 \geq 0, \Delta_2 \geq 0, \dots, \Delta_n \geq 0$: pos. semidefinite
$\Delta_1 \leq 0, \Delta_2 \geq 0, \dots$: neg. semidefinite

In ~~case~~ 3) and 4):

$\Delta_1, \Delta_2, \dots, \Delta_n \geq 0 \iff A$ positive semidefinite

$\Delta_1 \leq 0, \Delta_2 \geq 0, \Delta_3 \leq 0, \dots \iff A$ negative semidefinite

$\uparrow$

Δ_i means <u>all</u> principal minors of order i.

Ex: $A = \begin{pmatrix} 3 & 1 & 9 \\ 1 & 2 & 8 \\ 9 & 8 & 42 \end{pmatrix}$ symm.

$D_1 = \underline{3}$

$D_2 = \underline{5}$

$D_3 = 3 \cdot (84 - 64) - 1 \cdot (42 - 72) + 9 \cdot (8 - 18) = 60 + 30 - 90$

$= \underline{0}$

Concl: A may be positive semidefinite

All principal minors

choose one row and the same column

$\Delta_1 = 3, \ 2, \ 42 \geq 0$ $(1,1)$ $(2,2)$ $(3,3)$

$\Delta_2 = 5, \ \begin{vmatrix} 2 & 8 \\ 8 & 42 \end{vmatrix} = 20, \ \begin{vmatrix} 3 & 9 \\ 9 & 42 \end{vmatrix} = 126 - 81 = 45 \geq 0$

$\Delta_3 = 0 \geq 0$

chose two rows and the same two col's:

$\binom{12}{12}$ $\binom{13}{13}$ $\binom{23}{23}$

A is <u>positive semidefinite</u>

vi) Markov chains

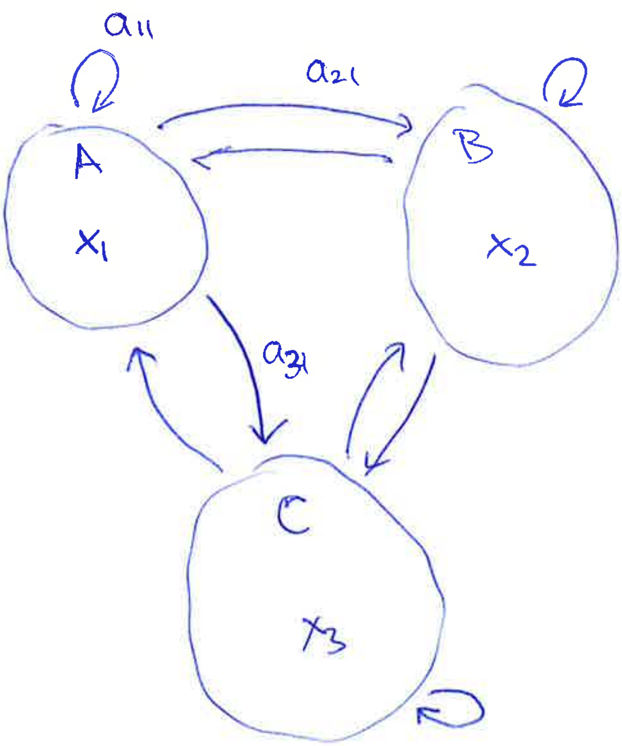
A $n \times n$-matrix (transition matrix) $\begin{cases} \text{all entries } a_{ij} \in [0,1] \\ \text{each column has sum} = 1 \end{cases}$

if $a_{ij} > 0$ then it is called <u>regular</u>

$$\boxed{\underline{x}_{n+1} = A \underline{x}_n} \longrightarrow \underbrace{\lim_{n\to\infty} \underline{x}_n = \lim_{n\to\infty} A^n \cdot \underline{x}_0}$$

long run equilibrium if the limit exists

Fact: A regular Markov chain has a long run equilibrium state $\underline{\underline{x}}$, and $\underline{x}$ is the unique eigenvector with $\lambda = 1$ with $x_1 + x_2 + \dots + x_n = 1$.



$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ state vector gives the share in each state.

$x_1, x_2, x_3 \geq 0$

$x_1 + x_2 + x_3 = 1$

$A = \begin{bmatrix} a_{11} & \cdot & \cdot \\ a_{21} & \cdot & \cdot \\ a_{31} & \cdot & \cdot \end{bmatrix}$

$a_{11} + a_{21} + a_{31} = 1$

$\underline{x}_0 = \begin{pmatrix} x_{10} \\ x_{20} \\ x_{30} \end{pmatrix} \to \underline{x}_1 = A \cdot \underline{x}_0$

$\to x_2 = A^2 \cdot \underline{x}_0 \to \dots \to \underline{x}$

long run equilibr. state

② Unconstrained optimization

Basic techniques: $\boxed{\text{a) Compute derivatives} \quad \text{b) Find Hessian matrices}}$ incl. e^x, $\ln x$

a) Stationary pts:

Fact 1:

A stationary pt for f is a pt such that $\boxed{f'_{x_1} = f'_{x_2} = \ldots = f'_{x_n} = 0}$

Fact 2:

If $\underline{x}^*$ is a local/global max/min, then $\underline{x}^*$ is a stationary pt.

Fact 3: Second derivative test

A stationary pt $\underline{x}^*$ can be classified as local max, local min or saddle pt using Hessian:

$H(f)(x^*)$

Hessian of f at $\underline{x}^*$

positive definite $\Longrightarrow$ x^* local min

negative definite $\Longrightarrow$ x^* local max

indefinite $\Longrightarrow$ $\underline{x}^*$ saddle pt

other cases : no conclusion

Ex: $f(x,y,z) = x^2 + y^2 + z^2 + z^3 + 2yz - 2x + 12y$

Stationary pts: $f'_x = 2x - 2 = 0$ $\quad \underline{x = 1}$

$f'_y = 2y + 2z + 12 = 0$

$f'_z = 2z + 3z^2 + 2y = 0$

$y + z + 6 = 0 \Rightarrow y = \underline{-6 - z}$

$2z + 3z^2 + 2\cdot(-6 - z) = 0$ $\quad 3z^2 - 12 = 0$

$z^2 = 4$

$z = \pm 2$

$z = 2,\ y = -8,\ x = 1$

$z = -2,\ y = -4,\ x = 1$

$$\boxed{\text{Stat. pts: } (x,y,z) = (1,-8,2),\ (1,-4,-2)}$$

Classification:

$$H(f) = \begin{array}{ccc} f''_{xx} & f''_{xy} & \\ \end{array}$$
$$H(f) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2+6z \end{pmatrix}$$

$\underline{(1,-8,2)}$: $H(f)(1,-8,2) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 14 \end{pmatrix}$ $\quad D_1 = 2$, $D_2 = 4$, $D_3 = 2\cdot 24 = 48$

positive defn. $\Rightarrow$ local min at $(1,-8,2)$

$(1,-4,-2)$: $H(f)(1,-4,-2)$

$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & -10 \end{pmatrix}$ $\quad D_1 = 2$, $D_2 = 4$, $D_3 = 2(-20-4) = -48$

indefinite $\Rightarrow$ saddle pt at $(1,-4,-2)$

ii) Convex / concave functions and global max/min

Fact 1:

f is convex $\iff$ $H(f)$ is positive semidefinite for all $\underline{x}$

f is concave $\iff$ $H(f)$ is negative semi definite —"—

Fact 2:

If f is concave, then any stationary pt is global max,

—"— convex —— " —— global min.

Ex: $f(x,y,z) = x^2 + y^2 + y^4 + yz - 1$

$$f'_x = 2x$$

$$f'_y = 2y + 4y^3 + z \qquad H(f) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2+12y^2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$f'_z = y$$

$D_1 = 2 > 0$ for all (x,y,z)

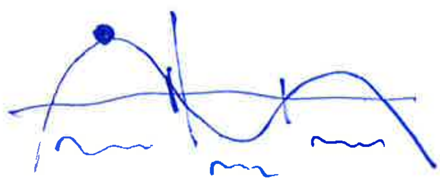
$D_2 = 2(2+12y^2) = 4 + 24y^2 > 0$

$D_3 = 2 \cdot (0 - 1) = -2 < 0$

$H(f)$ indefinite for all (x,y,z).

f is not convex, not concave

Even if f is not convex and not concave, it could of course still have global max/min.



iii) Envelope theorem

$f(x_1,..,x_n; a) = f(\underline{x}; a)$ function with parameter a

Consider the unconstrained optimization problem

$$\boxed{\max/\min \ f(\underline{x}; a)}$$

Assume that it has solution $\underline{x}^*(a)$ depending on a, and let $f^*(a) = f(\underline{x}^*(a))$ be the max/min value.

$$\boxed{\text{Envelope thm:} \quad \frac{df^*(a)}{da} = \frac{\partial f}{\partial a}(\underline{x}^*(a))}$$

this tells you how changing a will change the max/min value

this is how compute it.

Ex: $\min f(x,y;h) = hx^4 + y^4 + 4x^2 - (6+h)xy + 4y^2 - 3h$

$$\frac{\partial f}{\partial h} = x^4 - 3$$

$$\frac{df^*(h)}{dh} = (x^*(h))^4 - 3 - x^*(h)y^*(h)$$

i) For which values of h is f ~~convex~~ convex?

$f'_x = 4hx^3 + 8x - (6+h)y$

$f'_y = 4y^3 - (6+h)x + 8y$

$$H(f) = \begin{pmatrix} 12hx^2 + 8 & -(6+h) \\ -(6+h) & 12y^2 + 8 \end{pmatrix}$$

$D_1 = 12hx^2 + 8$ ← When $h \geq 0$, $D_1 \geq 0$ for all (x,y)

$D_2 = (12hx^2+8)(12y^2+8) - (6+h)^2$

$= 144hx^2 + 96hx^2 + 96y^2 + [64 - (6+h)^2]$ ← When $h \leq 2$, $D_2 \geq 0$ for all (x,y)

$\Delta_1 = 12y^2 + 8 > 0$ for all (x,y) ← Check the other principal minor of order 1 since $D_2 = 0$ at $(0,0)$ when $h=2$

Conclusion: When $0 \leq h \leq 2$ f is convex

ii) Find $x^*(h), y^*(h)$ when $h=0$:

$h=0 \Rightarrow f$ convex, so any stationary pt is global min.

Stationary pts: $h=0$

$8x - 6y = 0$

$4y^3 - 6x + 8y = 0$

$x = \frac{6y}{8} = 3y/4$

$4y^3 - 6\cdot(3y/4) + 8y = 0$

$4y^3 - \frac{18}{4}y + 8y = 0 \quad | \cdot 2$

$8y^3 - 9y + 16y = 0$

$8y^3 + 7y = 0$

$y(8y^2+7) = 0$

$y=0$ or $y^2 = -7/8$ (no sol'n)

$y=0 \Rightarrow x=0$

Stat. pts: $(x,y) = (0,0)$

This is global min for $h=0$, so $(x^*(0), y^*(0)) = (0,0)$

iii) If h increases from $h=0$, what happens with $f^*(h)$?

$f^*(0) = f(0,0) = -3h \quad \leftarrow$ min. value when $h=0$

$$\frac{df^*(h)}{dh} = \frac{\partial f}{\partial h}(x^*(h), y^*(h)) = (x^4 - xy - 3)\Big|_{x=x^*(h),\, y=y^*(h)}$$

$$= x^*(h)^4 - x^*(h)y^*(h) - 3$$

$$\frac{df^*(h)}{dh}\Big|_{h=0} = x^*(0)^4 - x^*(0)y^*(0) - 3 = 0^4 - 0\cdot 0 - 3 = -3$$

rate of change at $h=0$

The minimum value will decrease when h increases from $h=0$

What happens if h increases to $h > 0$?

Envelope thm:

$$\frac{d f^*(h)}{dh} = \frac{\partial f}{\partial h}\left(x^*(0), y^*(0); 0\right)$$

$$= \left(x^4 - xy - 3\right)(0, 0; 0) = \underline{-3}$$

Interpretation:

h incr. from 0 to 0.1

$$\rightsquigarrow f^*(0.1) \simeq \underbrace{f^*(0)}_{0} + \underbrace{h \cdot \frac{d f^*(h)}{dh}}_{0.1 \cdot (-3)} = \underline{-0.3}$$

Exam 12/2014, Q2

$$A = \begin{pmatrix} t & 1 & 1 \\ 1 & t & 1 \\ 1 & 1 & t \end{pmatrix}$$

a) $$|A| = \begin{vmatrix} t & 1 & 1 \\ 1 & t & 1 \\ 1 & 1 & t \end{vmatrix} = t \cdot \begin{vmatrix} t & 1 \\ 1 & t \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & t \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & t \\ 1 & 1 \end{vmatrix}$$

$$= t(t^2-1) - (t-1) + (1-t)$$

$$= t(t^2-1) - 2t + 2 = \underline{t^3 - 3t + 2}$$

$$= t(t^2-1) - 2(t-1)$$

$$= \underline{t(t+1)}(t-1) - 2(t-1) = (t-1)\cdot(t^2+t-2)$$

$$= (t-1)\cdot(t-1)(t+2) = \underline{(t-1)^2(t+2)}$$

$$A - \lambda I = \begin{pmatrix} t-\lambda & 1 & 1 \\ 1 & t-\lambda & 1 \\ 1 & 1 & t-\lambda \end{pmatrix} \underset{\uparrow}{=} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$\lambda = t-1$

$t - \lambda = t-(t-1)$

$= 1$

$$\text{rk}(A - (t-1)I) = \text{rk} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \underline{1}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

b) $\underline{t=8}$: $A = \begin{pmatrix} 8 & 1 & 1 \\ 1 & 8 & 1 \\ 1 & 1 & 8 \end{pmatrix}$

BI

A is symmetric, therefore it is diagonalizable.

<u>Eigenvalues</u>:

$$|A-\lambda I| = \begin{vmatrix} 8-\lambda & 1 & 1 \\ 1 & 8-\lambda & 1 \\ 1 & 1 & 8-\lambda \end{vmatrix} = 0$$

<u>Alt 1</u>: $\begin{vmatrix} t & 1 & 1 \\ 1 & t & 1 \\ 1 & 1 & t \end{vmatrix} = 0 \Leftrightarrow (t-1)^2(t+2) = 0$

$\begin{vmatrix} 8-\lambda & 1 & 1 \\ 1 & 8-\lambda & 1 \\ 1 & 1 & 8-\lambda \end{vmatrix} = 0 \Leftrightarrow (8-\lambda-1)^2 \cdot (8-\lambda+2) = 0$

$(7-\lambda)^2 \cdot (10-\lambda) = 0$

$\underline{\lambda_1 = \lambda_2 = 7, \ \lambda_3 = 10}$

<u>Alt 2</u>: $\lambda = 7$ is a solution $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$ $\quad rk\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = 1$

$\Downarrow$

<u>Why?</u> $\longrightarrow$ $\lambda = 7$ has multiplicity at least two.

$\lambda = 7$ eigenvalue

$\Rightarrow$ eigenvectors for $\lambda = 7$:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

y, z free

$x = -y-z$

$1 \le \underset{2}{\underset{\shortparallel}{\#\text{ free vars}}} \le \text{multiplicity of } \lambda$

$\underline{\lambda_1 = \lambda_2 = 7}$ $\quad \lambda_3 = ?$

$\lambda_1 + \lambda_2 + \lambda_3 = tr(A) = 24$

$7 + 7 + \lambda_3 = 24$

$\underline{\lambda_3 = 10}$

Att 3: $\quad A = \begin{pmatrix} 8 & 1 & 1 \\ 1 & 8 & 1 \\ 1 & 1 & 8 \end{pmatrix}$

$$\begin{vmatrix} 8-\lambda & 1 & 1 \\ 1 & 8-\lambda & 1 \\ 1 & 1 & 8-\lambda \end{vmatrix} = 0$$

$$a^2-b^2=(a-b)(a+b)$$

$$(8-\lambda)\cdot\left[(8-\lambda)^2-1\right]-1\cdot\left[8-\lambda-1\right]+1\cdot\left[1-(8-\lambda)\right]=0$$

$$(8-\lambda)\cdot(8-\lambda-1)(8-\lambda+1) + \underline{1-(8-\lambda)} + \underline{1-(8-\lambda)}$$

$$=(8-\lambda)\ (8-\lambda-1)(8-\lambda+1) + 2\cdot(1-8+\lambda)$$

$$=(8-\lambda)\ (\underline{7-\lambda})(9-\lambda) + 2\,(\underline{-7+\lambda})$$

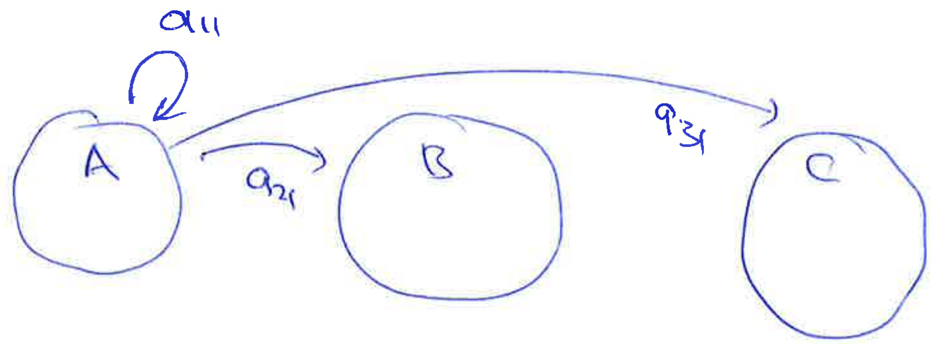
$$=(\lambda-7)\cdot\left[\underbrace{(8-\lambda)(9-\lambda)(-1)+2}\right]=0$$

$\underline{\lambda=7}$ $\quad$ or $\quad -(\lambda^2-17\lambda+72)+2=0$

$$\lambda^2-17\lambda+72=2$$
$$\lambda^2-17\lambda+70=0$$

$$\underline{\lambda=7},\ \underline{\lambda=10}$$

c)



$$a_{11} = 8 \cdot a_{21} = 8 \cdot a_{31}$$

$$\underline{a_{21} = s}$$

$$8s + s + s = 1$$
$$10s = 1$$
$$s = \tfrac{1}{10}$$

$$a_{11} = 8s \qquad a_{21} = s \qquad a_{31} = s$$

$$A = \begin{pmatrix} 8s & s & s \\ s & 8s & s \\ s & s & 8s \end{pmatrix} = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.8 & 0.1 \\ 0.1 & 0.1 & 0.8 \end{pmatrix}$$

<u>Long run equilibrium state:</u>

$\underline{\lambda = 1}$:
$$\begin{pmatrix} 0.8-1 & 0.1 & 0.1 \\ 0.1 & 0.8-1 & 0.1 \\ 0.1 & 0.1 & 0.8-1 \end{pmatrix} = \begin{pmatrix} -0.2 & 0.1 & 0.1 \\ 0.1 & -0.2 & 0.1 \\ 0.1 & 0.1 & -0.2 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ \cancel{0} & \cancel{3} & \cancel{-3} \end{pmatrix} \leftarrow \begin{pmatrix} +1 & +1 & -2 \\ +1 & -2 & +1 \\ -2 & +1 & +1 \end{pmatrix} \leftarrow \begin{pmatrix} -2 & +1 & +1 \\ +1 & -2 & +1 \\ +1 & +1 & -2 \end{pmatrix}$$

$$x + y - 2z = 0 \qquad x + z - 2z = x - z = 0 \qquad \underline{x = z}$$
$$-3y + 3z = 0 \qquad \underline{y = z}$$

$$\underline{x} = \begin{pmatrix} z \\ z \\ z \end{pmatrix} = z \cdot \underline{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$$

$$\boxed{\text{State vector: } \quad x + y + z = 1} \qquad x, y, z \geq 0$$

$$3z = 1 \qquad z = 1/3$$

<u>Answer:</u> $\frac{120}{3} = \underline{\underline{40}}$

$$\underline{x} = \underline{\begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}}$$

Question I. $f(x,y,z) = x^4 + y^2 - xz + z^4$

BI

a) $f'_x = 4x^3 - z$

$f'_y = 2y$

$f'_z = -x + 4z^3$

$$H(f) = \begin{pmatrix} 12x^2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 12z^2 \end{pmatrix}$$

b) $$\left.\begin{aligned} f'_x &= 4x^3 - z = 0 \\ f'_y &= 2y = 0 \\ f'_z &= -x + 4z^3 = 0 \end{aligned}\right\}$$

Stationary pts:

$2y = 0 \Rightarrow \underline{y = 0}$

$4x^3 - z = 0 \Rightarrow \underline{z = 4x^3}$

$\Rightarrow -x + 4\cdot(4x^3)^3 = 0$

$-x + 4\cdot 64x^9 = 0$

$x\cdot(-1 + 256\cdot x^8) = 0$

$\underline{x = 0}$ or $256x^8 = 1$

$256 = 4^4 = 2^8$

$x^8 = \frac{1}{256}$

$x = \pm(1/256)^{1/8}$

$\underline{\underline{x = \pm 1/2}}$

$\Downarrow$

$\underline{z = \pm 1/2}$

$x = 0$

$\Downarrow$

$\underline{z = 0}$

$\Rightarrow$ Stationary pts:

$x = 0,\ y = 0,\ z = 0$

$x = 1/2,\ y = 0,\ z = 1/2$

$(0,0,0)$

$(\frac{1}{2}, 0, \frac{1}{2})$

$(-\frac{1}{2}, 0, -\frac{1}{2})$

Classify:

$(0,0,0)$: $H(f)(0,0,0) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix}$

$D_1 = 0$ $\Delta_1 = 0, 2, 0$

$D_2 = 0$

$D_3 = -1\cdot 2 = -2$

indefinite $\Rightarrow$ $(0,0,0)$ saddle pt.

$\underline{x} = (1/2, 0, 1/2)$:

$x = (-1/2, 0, -1/2)$:

$12x^2$

$$H(f)(\underline{x}) = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix}$$

$12z^2$

$D_1 = 3 \quad D_2 = 6 \quad D_3 = 2\cdot(9-1) > 0$

pos. defn. $\Rightarrow$ local min $(1/2, 0, 1/2)$ and $(-1/2, 0, -1/2)$

Concl: $(0,0,0)$ saddle pt.

$(\pm 1/2, 0, \pm 1/2)$ local min

c)

$$H(f) = \begin{pmatrix} 12x^2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 12z^2 \end{pmatrix}$$

$\leftarrow$ Hessian of f for a general pt. (x,y,z)

convex $\Leftrightarrow$ $H(f)(x,y,z)$ pos. semidefn. for all (x,y,z)

concave $\Leftrightarrow$ — " — neg. — " —

Alt: Since $H(f)(0,0,0)$ is indefinite from b) f is not convex, not concave.

Alt: $D_1 = 12x^2 \geq 0$

$D_2 = 24x^2 \geq 0$

$D_3 = 2\cdot(144x^2z^2 - 1)$ $\leftarrow$ can be both pos. and neg.

$x=0, z=0$: -2

$x=1, z=1$: $2\cdot(144-1) > 0$

f is not concave, not convex.