LECTURE 12

GRA 6035 MATHEMATICS

BI

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Plan:

1. Review: Second order linear differential equations
2. Difference equations
3. Stability

Reading:

[MET] 23.2

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1 Differential equations

First order:

- seperable $\quad y' = f(y) \cdot g(t)$
- linear $\quad y' + a(t)y = b(t)$
- exact $\quad p(y,t) + q(y,t)y' = 0$, $\quad p = \frac{\partial h}{\partial t}$, $\quad q = \frac{\partial h}{\partial y}$

Second order:

- linear $\quad y'' + ay' + by = f(t)$

$y = y_h + y_p$

Ex: $y'' - 11y' + 28y = t + 4$

$$y = y_h + y_p = C_1 e^{4t} + C_2 e^{7t} + \frac{1}{28}\left(t + \frac{123}{28}\right)$$

y_h: $y'' - 11y' + 28y = 0$

$r^2 - 11r + 28 = 0$

$r = \frac{11 \pm \sqrt{11^2 - 4 \cdot 28}}{2}$

$= \frac{11 \pm \sqrt{9}}{2} = 7, 4 \Rightarrow y_h = C_1 e^{4t} + C_2 e^{7t}$

y_p: $\quad y'' - 11y' + 28y = t+4$

$f(t) = t+4$

$f'(t) = 1$

$f''(t) = 0$

Guess: $y_p = At+B$

$y = At+B$

$y' = A$

$y'' = 0$

$$0 - 11\cdot A + 28(At+B) = t+4$$

$$\underbrace{(28A)}_{=1}t + \underbrace{(28B - 11A)}_{=4} = t+4$$

$28A = 1$

$28B - 11A = 4$

$A = 1/28$

$28B - 11(1/28) = 4$

$28B = 4 + 11/28 = \frac{123}{28}$

$B = \frac{123}{28^2}$

$\Downarrow$

$$y_p = At+B = \frac{1}{28}t + \frac{123}{28^2} = \frac{1}{28}\left(t + \frac{123}{28}\right)$$

Hint: If your initial guess for y_p doesn't work, try to multiply your guess with t.

If you have a first order linear equation of the form

$$\boxed{y' + ay = f(t)} \qquad \begin{cases} a: \text{constant} \\ f(t): \text{expression in } t \end{cases}$$

then we may use the method from second order linear eqns instead of integrating factor.

Ex: $y' + \underset{\substack{\uparrow \\ \text{constant}}}{4}y = t \longleftrightarrow$ first order linear

$$y = y_h + y_p = \underline{\underline{C \cdot e^{-4t} + \left(\tfrac{1}{4}t - \tfrac{1}{16}\right)}}$$

y_h: $y' + 4y = 0$

$r + 4 = 0$

$\underline{r = -4} \longrightarrow y_h = C \cdot e^{-4t}$

y_p: $y' + 4y = t \longrightarrow f(t) = t \rightarrow$ Guess: $y_p = At + B$

$f' = 1$

$\cancel{f'' = 0}$

$y = At + B$

$y' = A$

$A + 4(At + B) = t$

$\underbrace{(4A)}_{1}t + \underbrace{(4B + A)}_{0} = t$

$4A = 1 \Rightarrow A = \frac{1}{4}$

$4B + A = 0 \Rightarrow$

$4B = -\frac{1}{4}$

$B = -\frac{1}{16}$

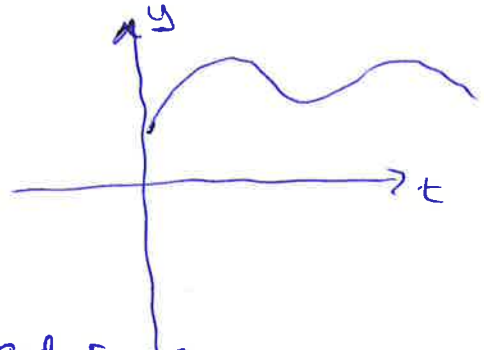
$y_p = At + B = \underline{\frac{1}{4}t - \frac{1}{16}}$

Stability: Second order linear eqns.

$$y'' + ay' + by = f(t) \implies y = y_h + y_p$$

if $a^2 - 4b > 0$:

$$= C_1 e^{r_1 t} + C_2 e^{r_2 t} + y_p(t)$$



What happens as $t \to \infty$?

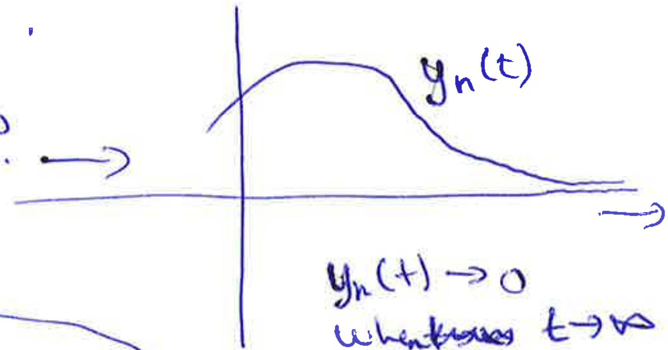
$$\lim_{t\to\infty} y_h(t) = \lim_{t\to\infty}\left(C_1 e^{r_1 t} + C_2 e^{r_2 t}\right)$$

$$= \begin{cases} 0 & , \ r_1 < 0 \text{ and } r_2 < 0 \\ * & , \ \text{otherwise} \end{cases}$$

$\uparrow$ (besides ∞)

The diff. eqn. is called globally asymptotically stable if $r_1 < 0, r_2 < 0$.

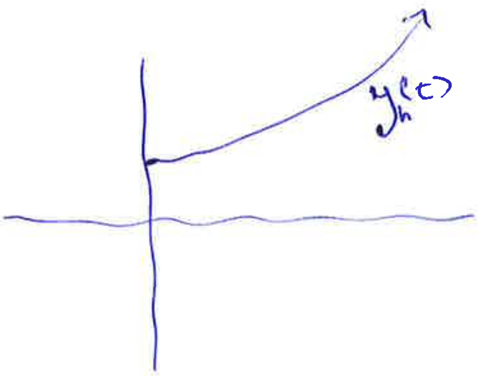
globally asympt. stable $\longrightarrow$



$y_h(t) \to 0$ whenever $t \to \infty$ no matter which initial conditions you have.

Ex: $$y = \underset{\downarrow\ \infty}{C_1 e^{4t}} + \underset{\downarrow\ \infty}{C_2 e^{7t}} + \underset{\downarrow\ \infty}{y_p}$$

$$y_p = \frac{1}{28}\left(t + \frac{123}{28}\right)$$



Ex: $y'' + 3y' - 4y = -12$

$$y = \underbrace{C_1 e^{-4t} + C_2 e^{t}}_{y_h} + \underbrace{3}_{y_p}$$

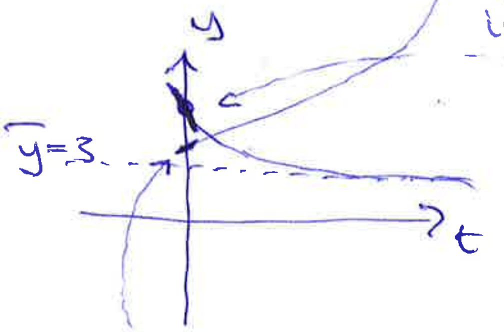
$$\lim_{t\to\infty} y_h(t) = C_1 \cdot \lim_{t\to\infty} e^{-4t} + C_2 \cdot \lim_{t\to\infty} e^{t}$$

$$= C_1 \cdot 0 + C_2 \cdot \infty$$

$$= \begin{cases} 0, & C_2 = 0 \\ \pm\infty, & C_2 \neq 0 \end{cases}$$

not globally asympt. stable

initial cond.



initial cond

$$\left.\begin{array}{l} y(0) = 6 \\ y'(0) = -12 \end{array}\right\} \Rightarrow \begin{array}{l} C_1 = 3 \\ C_2 = 0 \end{array}$$

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \left(3e^{-4t} - 3\right)$$

$$= \underline{-3}$$

Stability means there is long term equilibrium with this initial value.

$$\left.\begin{array}{l} y(0) = 4 \\ y'(0) = 1 \end{array}\right| \begin{array}{l} C_1 = 0 \\ C_2 = 1 \end{array}$$

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \left(e^{t} + 3\right) = \infty$$

not stable

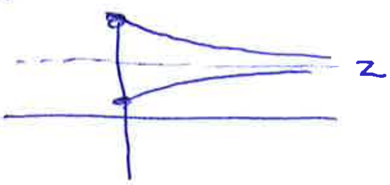
Ex: $y'' + 3y' + 2y = 4$

$$y = C_1 e^{-2t} + C_2 e^{-t} + 2$$

$$\lim_{t\to\infty} y(t) = C_1 \cdot 0 + C_2 \cdot 0 + 2$$

$$= \underline{2}$$

globally asympt. stable



Long run equilibrium = 2 no matter what the initial cond. are.

(2) Difference equations

differential equations — change in continuous time

difference equations — change in discrete time

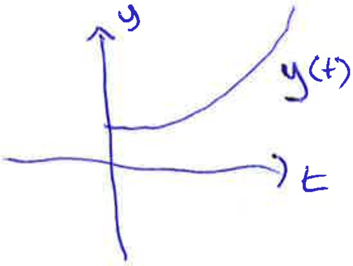
Ex: Interest is added to a bank account

$y(t)$: balance at time t

$$y' = r \Rightarrow y = B_0 e^{rt}$$

← r is the cont. interest rate

cont. time = differential eqn.



y_t : balance at time t

$$\underbrace{y_{t+1} - y_t}_{\text{change}} = r \cdot y_t$$

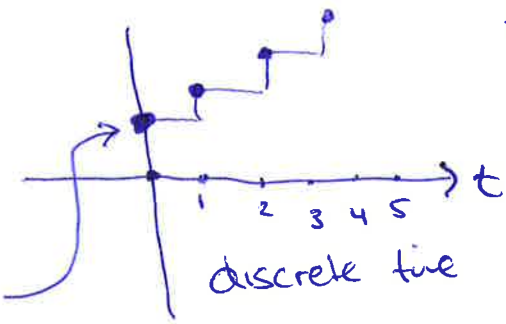
← r is the interest rate per term

$$y_{t+1} - (1+r) y_t = 0$$

$$y_{t+1} = y_t + r y_t$$

$$y_{t+1} = (1+r) y_t$$

$$\underline{y_t = B_0 \cdot (1+r)^t}$$



discrete time

$y_0 = B_0$

$y_1 = B_1 = B_0 \cdot (1+r)$

$y_2 = B_2 = B_0 \cdot (1+r)^2$

$\vdots$

Defn: A <u>first order</u> difference equation is an equation involving

$$\underbrace{y_{t+1},\ y_t}_{\text{difference in indices is } \underline{1}} \qquad \boxed{y_{t+1} = F(y_t, t)}$$

Ex: $y_{t+1} + 2y_t = 4 \to y_{t+1} = 4 - 2y_t$

$y_{t+1} = 4y_t - t^2 \to y_{t+1} - 4y_t = -t^2$

A <u>solution</u> of a difference equation is a sequence $(y_t)_{t=0,1,2,\ldots}$ that fits in the difference equation. It is usually given by a formula (closed form), for instance

$$\boxed{y_t = (-2)^t + 4}$$

$\Updownarrow$

$y_0 = 5 \quad y_1 = 2 \quad y_2 = 8 \quad y_3 = -4 \quad \ldots$

Linear first order difference equations (with constant coeffs.)

$y_{t+1} + a y_t = f_t$

a : constant
f_t : expression in t

$$\underline{y_{t+1} + a \cdot y_t = f_t}$$

Homogeneous if $f_t = 0$
Inhomogeneous otherwise

a) <u>Homogeneous case:</u>

$$y_{t+1} + a y_t = 0$$

$$\underbrace{y_{t+1} - y_t}_{\text{change}} = y_t$$

$a = -2$

<u>Ex:</u> $y_{t+1} - 2y_t = 0$

$y_{t+1} = 2y_t$

$y_0 \longrightarrow y_1 = 2y_0$

$y_2 = 2y_1 = 2(2y_0) = 2^2 \cdot y_0$

$y_3 = 2y_2 = 2 \cdot (2^2 y_0) = 2^3 y_0$

$\vdots$

$y_t = 2^t \cdot y_0$

<u>Solution:</u> $\boxed{y_t = 2^t \cdot y_0}$

$y_{t+1} - 2y_t = 0, \quad y_0 = 1$

$\Downarrow$

$y_t = 2^t \cdot y_0 = 2^t \cdot 1$

$\underline{\underline{y_t = 2^t}}$

<u>In general:</u> $y_{t+1} + a y_t = 0$

$y_{t+1} = (-a) \cdot y_t$

$y_1 = (-a) y_0$
$y_2 = (-a)^2 y_0$
$\vdots$
$y_t = (-a)^t \cdot y_0$

<u>Solution:</u> $\underline{y_t = (-a)^t \cdot y_0}$

<u>Characteristic eqn:</u>

$r + a = 0$

$r = -a$

<u>Solution:</u> $y_t = r^t \cdot y_0 = \underline{r^t \cdot C}$

$$y_{t+1} + a \cdot y_t = 0$$

← linear first order homogeneous difference eqn.

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Characteristic eqn:

$$r + a = 0$$

$$r = -a \longrightarrow y_t = C \cdot r^t = C \cdot (-a)^t$$

b) Inhomogeneous case:

$$\boxed{y_{t+1} + a y_t = f_t}$$

Ex: $y_{t+1} - y_t = t^2$

y_0

$y_1 = y_0 + 0 = y_0$

$y_2 = y_1 + 1 = y_0 + 1$

$y_3 = y_2 + 4 = y_0 + 5$

$\vdots$

$y_t = ?$

① Superposition principle:

$$y_t = y_t^h + y_t^p$$

↑ general homog. solution

↑ particular inhom. solution

y_t^h: $y_{t+1} - y_t = 0$

r^t not e^{rt}

discr. time

cont. time

$$r - 1 = 0$$

$$r = 1 \longrightarrow y_t^h = C \cdot 1^t = C$$

② Use educated guess to find y_t^p

y_t^p: $y_{t+1} - y_t = t^2$

$f_t = t^2$

$f_{t+1} = (t+1)^2 = t^2 + 2t + 1$

Guess: $y_t = At^2 + Bt + C$

$y_{t+1} = A \cdot (t+1)^2 + B(t+1) + C$

$= At^2 + (2A+B)t + (A+B+C)$

$$\left[At^2 + (2A+B)t + (A+B+C)\right] - \left[At^2 + Bt + C\right] = t^2$$

New guess: $y_t = (At^2 + Bt + C)\cdot t$

$$= At^3 + Bt^2 + Ct$$

$$y_{t+1} = A(t+1)^3 + B(t+1)^2 + C(t+1)$$

$$= A\cdot(t^3 + 3t^2 + 3t + 1) + B(t^2 + 2t + 1) + C\cdot(t+1)$$

$$= At^3 + (3A+B)t^2 + (3A+2B+C)t + (A+B+C)$$

$y_{t+1} - y_t = t^2$!

$$\left[\cancel{At^3} + (3A+B)t^2 + (3A+2B+C)t + (A+B+C)\right] - \left[\cancel{At^3} + Bt^2 + Ct\right] = t^2$$

$$\underbrace{(3A)}_{=1} t^2 + \underbrace{(3A+2B)}_{=0} t + \underbrace{(A+B+C)}_{=0} = t^2$$

$\Downarrow$ $\quad\Downarrow\quad$ $\Downarrow$

$A = 1/3$

$3\cdot 1/3 + 2B = 0$

$1 + 2B = 0$

$B = -1/2$

$\frac{1}{3} + \left(-\frac{1}{2}\right) + C = 0$

$C = \frac{1}{2} - \frac{1}{3} = \frac{3}{6} - \frac{2}{6} = \frac{1}{6}$

$$y_t^p = At^3 + Bt^2 + Ct = \frac{1}{3}t^3 - \frac{1}{2}t^2 + \frac{1}{6}t$$

$$y_t = y_t^h + y_t^p = C + \frac{1}{3}t^3 - \frac{1}{2}t^2 + \frac{1}{6}t$$

$t=0$: $\quad y_0 = C + 0 \Rightarrow y_0 = C$

$$y_t = y_0 + \frac{1}{3}t^3 - \frac{1}{2}t^2 + \frac{1}{6}t$$

Second order linear difference equations

$$y_{t+2} + a y_{t+1} + b y_t = f_t \qquad \begin{cases} a, b: \text{constants} \\ f_t: \text{expression in } t \end{cases}$$

Second order: Involves $\underbrace{y_{t+2}, y_{t+1}, y_t}_{\text{difference of indices} = 2}$

change $\quad \underbrace{y_{t+1} - y_t}$ replaces y'

"change of change" $\quad (y_{t+2} - y_{t+1}) - (y_{t+1} - y_t)$

$$= \underbrace{y_{t+2} - 2y_{t+1} + y_t}_{\text{replaces } y''}$$

Solution method:

$$y_{t+2} + a y_{t+1} + b y_t = f_t$$

Superposition principle: $\quad y_t = y_t^h + y_t^p$

y_t^p: Exactly in the same way as first order difference equations

Look at f_t but also f_{t+1}, f_{t+2}

y_t^h: Look at homogeneous equation:

$$y_{t+2} + a y_{t+1} + b y_t = 0$$

Char. eqn: $r^2 + ar + b = 0$

$$r = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

$a^2-4b>0$: $\quad r_1 \neq r_2$ two roots

$$y_t^h = c_1 \cdot r_1^t + c_2 \cdot r_2^t$$

$$r = \frac{-a \pm \sqrt{a^2-4b}}{2}$$

$a^2-4b=0$: $\quad r$ double root $\quad (r = -a/2)$

$$y_t^h = c_1 \cdot r^t + c_2 t \cdot r^t$$
$$= (c_1 + c_2 t) \cdot r^t$$

$a^2-4b<0$:

there is a solution that can be expressed in terms of sin/cos

$$y_t = (\sqrt{b})^t \cdot \left(c_1 \cos(\theta t) + c_2 \sin(\theta t)\right)$$
where $\theta = \cos^{-1}(-a/2\sqrt{b})$

Ex: $\quad y_{t+2} - 3y_{t+1} + 2y_t = 5$

$$y_t = y_t^h + y_t^p = c_1 \cdot 2^t + c_2$$

y_t^h: $\quad y_{t+2} - 3y_{t+1} + 2y_t = 0$

$$r^2 - 3r + 2 = 0$$
$$r = 2, \quad r = 1$$
$$y_t^h = c_1 \cdot 2^t + c_2 \cdot 1^t$$
$$= c_1 \cdot 2^t + c_2$$

y_t^p: $\quad f_t = 5 \quad f_{t+1} = 5 \quad f_{t+2} = 5 \qquad y_t = A$

$$1A - 3A + 2A = 5$$
$$0 = 5$$

New guess:
$$\begin{cases} y_t = At \\ y_{t+1} = A\cdot(t+1) = At + A \\ y_{t+2} = A\cdot(t+2) = At + 2A \end{cases}$$

$$y_{t+2} - 3y_{t+1} + 2y_t = 5$$

$$(At + 2A) - 3(At + A) + 2(At) = 5$$

$$(\underbrace{A - 3A + 2A}_{=0})t + (\underbrace{2A - 3A}_{=-A}) = 5$$

$$0 \cdot t - A = 5$$

$$A = -5 \longrightarrow y_t^p = -5t$$

$$y_t = y_t^h + y_t^p = C_1 \cdot 2^t + C_2 - 5t$$