LECTURE 11

GRA 6035

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Plan:

① Exact differential equations

② Second order linear differential equations.

Reading:

[MET] 24.1-24.3, (24.4-24.6),

① Exact differential equations

First order differential equations:

ⓐ Seperable diff. eqn:

$$y' = f(y)\cdot g(t) \Rightarrow \frac{1}{f(y)} y' = g(t)$$

$$\int \frac{1}{f(y)}\, dy = \int g(t)\, dt$$

ⓑ Linear first order diff. eqn:

$$y' + a(t)\cdot y = f(t) \Rightarrow \text{int. factor: } u = e^{\int a(t)dt}$$

$$(y\cdot u)' = f(t)\cdot u$$

$$y\cdot u = \int f(t)\cdot u(t)\, dt$$

$$y = \frac{1}{u(t)}\cdot \int f(t)\, u(t)\, dt$$

c). Exact differential equations

Ex: $$\underbrace{1+t\cdot y^2}_{p}+\overbrace{t^2 y}^{q}\cdot y'=0$$

$$t^2y\cdot y' = -(1+ty^2)$$

$$y' = -\frac{1+ty^2}{t^2y}$$

← not separable
← not linear

Is it exact?

Defn. of exact diff. eqn.

$$p(y,t) + q(y,t)\cdot y' = 0$$

This equation is exact if there is a function $h(y,t)$ s.t.

$$\frac{\partial h}{\partial t} = p \qquad \frac{\partial h}{\partial y} = q$$

Ex: i) $p = 1+ty^2 = h'_t$ ⟺ exact
ii) $q = t^2y = h'_y$

i) $1+ty^2 = h'_t \Rightarrow h = t + y^2\cdot\frac{1}{2}t^2 + C(y)$

$$\boxed{\begin{aligned} h &= \int 1+ty^2\,dt \\ &= t+y^2\cdot\tfrac{1}{2}t^2 + C(y)\end{aligned}}$$

ii) $t^2y = h'_y$: $h \to h'_y = 0 + \cancel{\tfrac{1}{2}}t^2\cdot\cancel{2}y + C'(y)$

$t^2y = t^2y + C'(y)$

ok if $C'(y)=0$

can choose $C(y)=0$

Conclusion: $h = t+\frac{1}{2}y^2t^2$ satisfies i) and ii)

the diff.-eqn. is exact.

This means that the general solution of the dif. equ. is

$$h(y,t) = C$$

$$t + \frac{1}{2}y^2t^2 = C$$

$$\frac{\cancel{\frac{1}{2}}y^2\cancel{t^2}}{\cancel{\frac{1}{2}t^2}} = \frac{C-t}{\frac{1}{2}t^2}$$

$$y^2 = \frac{C-t}{\frac{1}{2}t^2} \cdot \frac{2}{2} = \frac{2(C-t)}{t^2}$$

$$y = \pm\sqrt{\frac{2(C-t)}{t^2}}$$

Summary:

$p(y,t) + q(y,t)\cdot y' = 0$ is exact

$\Updownarrow$

there is a $h = h(y,t)$ s.t.

i) $h'_t = p$

ii) $h'_y = q$

In that case, the general solution is $\boxed{h(y,t) = C}$

Why? If it is exact, then it can by defn. be written as

$$\underbrace{\frac{\partial h}{\partial t} + \frac{\partial h}{\partial y}\cdot\frac{dy}{dt}}_{\text{total derivative of } h} = 0$$

$\frac{dh}{dt}$

Ex: $h(y,t) = t + \frac{1}{2}t^2y^2$

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Partial derivatives:

$$\left.\begin{aligned}\frac{\partial h}{\partial t} &= 1 + ty^2 \\ \frac{\partial h}{\partial y} &= t^2y\end{aligned}\right\}\begin{aligned}&\leftarrow \text{consider } y \text{ const.}\\ &\leftarrow \text{consider } t \text{ const.}\end{aligned}$$

Total derivative:

$$\frac{dy^2}{dt} = \frac{dy^2}{dy}\cdot\frac{dy}{dt}$$

$$\frac{dh}{dt} = \left(t + \frac{1}{2}t^2y^2\right)'$$

$$= 1 + \frac{1}{2}\left(2t\cdot y^2 + t^2\cdot 2y\cdot y'\right)$$

$$= 1 + ty^2 + t^2y\cdot y'$$

Relationship between total and partial derivative:

$$\frac{dh}{dt} = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial y}\cdot\frac{dy}{dt}$$

$$= \frac{\partial h}{\partial t} + \frac{\partial h}{\partial y}\cdot y' \longleftarrow \text{Exact diff. eqn.: } = 0$$

$$\Updownarrow$$

$$\frac{dh}{dt} = 0$$

$$\Updownarrow$$

$$h = C$$

Comments:

1) When you find h, it doesn't matter if you choose

$$h = t + \frac{1}{2}t^2y^2 \quad \text{or} \quad h = t + \frac{1}{2}t^2y^2 + C$$

since

$$h = t + \frac{1}{2}t^2y^2 + C_1 = C_2$$

$$t + \frac{1}{2}t^2y^2 = C_2 - C_1$$

2) $p(y,t) + q(y,t)\, y' = 0$ is exact

$\Uparrow$

$$\boxed{\frac{\partial p}{\partial y} = \frac{\partial q}{\partial t}}$$

(To check if it is exact or not)

$\Updownarrow$

$p = \frac{\partial h}{\partial t}$ and $q = \frac{\partial h}{\partial y}$ for some fn. h

To check if it is exact or not, and to find h if it is exact.

Ex: $\underbrace{3t^2 + y^2}_{p} + \underbrace{(2ty - 2)}_{q} y' = 0$, $y(1) = 2$

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial y} \cdot \frac{dy}{dt} = 0$$

$$\boxed{\begin{aligned} 3t^2 + y^2 &= h'_t \\ 2ty - 2 &= h'_y \end{aligned}}$$

$h = \underline{t^3 + y^2 t + C(y)}$

put in h in

$h'_y = 0 + 2yt + C'(y)$

$= 2yt + C'(y)$

$2ty - 2 = 2yt + C'(y)$

$-2 = C'(y)$

$C(y) = -2y$

Solution: $h = t^3 + y^2 t - 2y$

Solution of diff. eqn:

$t^3 + y^2 t - 2y = C$

$1 + \cancel{4} - \cancel{4} = C$

$\underline{C = 1}$

$y(1) = 2$

$t = 1$, $y = 2$

$$t^3 + y^2 t - 2y = 1$$

$$t \cdot y^2 - 2 \cdot y + (t^3 - 1) = 0$$

$$y = \frac{2 \pm \sqrt{4 - 4t(t^3-1)}}{2t}$$

$y(1) = 2$

$$y = \frac{1}{t} \pm \frac{1}{t}\sqrt{1 - t(t^3-1)}$$

$$y(1) = \frac{1}{1} \pm \frac{1}{1} \cdot \sqrt{1 - 1 \cdot (1-1)} = 1 \pm 1 = 2$$

$$y = \frac{1}{t} + \frac{1}{t}\sqrt{1 - t(t^3-1)}$$

2 Second order linear differential equations

$$y'' + a(t) \cdot y' + b(t) y = f(t)$$ ← second order linear diff. eqn.

$$y'' + a y' + b y = f(t)$$ ← second order linear diff. eqn. with constant coeffs.

a, b : constants
$f(t)$: function in t

i) Homogeneous: $f(t) = 0$
ii) Inhomogeneous: $f(t) \neq 0$

Ex: $$y'' = 6t - 2$$

(easy case with $a=b=0$)

$$y' = 3t^2 - 2t + C$$

$$y = t^3 - t^2 + Ct + D$$

Second order diff. eqn.

$\Downarrow$

general solution with two undetermined coeffs.

$$y'' = 6t - 2, \quad \boxed{\begin{array}{l} y(0) = 1 \\ y'(0) = 0 \end{array}}$$

This means that we need two initial conditions to determine C, D.

i) The homogeneous case:

$$y'' + ay' + by = 0$$

Ex: $y'' - 3y' + 2y = 0$

Method: Characteristic equation

$y'' + ay' + by = 0 \rightsquigarrow r^2 + ar + b = 0$

diff. eqn. — char. eqn.

$$r = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

char. roots

Solution of diff. eqn: $\longleftarrow$

a) $y = C_1 \cdot e^{r_1 t} + C_2 e^{r_2 t}$ — a) Two roots $r_1 \neq r_2$ when $a^2 - 4b > 0$

b) $y = (C_1 + C_2 t) e^{rt}$ — b) One double root r when $a^2 - 4b = 0$

c) $y = e^{\alpha t} \cdot (C_1 \cdot \cos \beta t + C_2 \cdot \sin \beta t)$ — c) No roots when $a^2 - 4b < 0$

where $\alpha = -\frac{a}{2}$

$$\beta = \frac{\sqrt{4b - a^2}}{2}$$

Ex: $y'' - 3y' + 2y = 0 \longrightarrow r^2 - 3r + 2 = 0$

$$r = \frac{3 \pm \sqrt{9 - 4 \cdot 2}}{2} = \frac{3}{2} \pm \frac{1}{2} = 2, 1$$

$r_1 = 2 \quad r_2 = 1$

$$y = C_1 \cdot e^{2t} + C_2 \cdot e^{t}$$

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Ex: $y'' - 4y' + 4y = 0 \rightsquigarrow r^2 - 4r + 4 = 0$

$$r = \frac{4 \pm \sqrt{16 - 4\cdot 4}}{2}$$

$$= 2 \quad \text{double root}$$

$$r_1 = 2 \qquad r_2 = 2$$

$$y = c_1 e^{2t} + c_2 t e^{2t} = \underline{(c_1 + c_2 t) e^{2t}} \longleftarrow$$

Ex: $y'' + y = 0 \rightsquigarrow r^2 + 1 = 0$

$$r = \frac{0 \pm \sqrt{0 - 4\cdot 1}}{2} = \frac{0}{2} \pm \frac{\sqrt{-4}}{2} = 0 \pm \frac{\sqrt{4}}{2}\cdot\sqrt{-1} = \underset{\alpha}{\underline{0}} \pm \underset{\beta}{\underline{1}}\cdot\sqrt{-1}$$

$$y = e^{0\cdot t}\cdot\left(c_1 \cos(1\cdot t) + c_2 \cdot \sin(1\cdot t)\right)$$

$$y = \underline{\underline{c_1 \cos(t) + c_2 \cdot \sin(t)}}$$

Why: $\boxed{y'' + ay' + by = 0}$

i) Expect solution: $y = \underline{c_1 \cdot y_1(t) + c_2 \cdot y_2(t)}$

need to find <u>two</u> different solution $\underline{y_1, y_2}$

ii) Guess solution $y = e^{rt}$:

$$\left.\begin{aligned} y &= e^{rt} \\ y' &= r\cdot e^{rt} \\ y'' &= r^2 e^{rt} \end{aligned}\right\} \quad \begin{aligned} &r^2 e^{rt} + a\cdot r e^{rt} + b\cdot e^{rt} = 0 \\ &e^{rt}\cdot(r^2 + ar + b) = 0 \\ &\underline{r^2 + ar + b = 0} \\ &\text{char. eqn.} \end{aligned}$$

e^{rt} is a solution of diff. eqn. $\Longleftrightarrow$ r is a solution of the cha. eqn. $\underline{r^2 + ar + b = 0}$

In the case $r_1 = r_2$, we need another solution since $e^{r_1 t} = e^{r_2 t}$

We can use $e^{r_1 t}$ and $t\cdot e^{r_1 t}$

(see problem in workbook)

ii) Inhomogeneous case:

$$y'' + ay' + by = f(t)$$

Ex: $y'' - 4y' + 7y = 12te^t$

Superposition principle:

$y = y_h + y_p$

The general solution of the inhomogeneous equation
$$y'' + ay' + by = f(t)$$
is $y = y_h + y_p$, where

y_h: general solution of the homogeneous equation
$$y'' + ay' + by = 0$$

y_p: a particular solution of the inhomogeneous equation

Ex: $y'' - 4y' + 7y = 3 \leftarrow f(t) = 3$

$$y = y_h + y_p = e^{2t}\left(C_1 \cos(\sqrt{3}t) + C_2 \sin(\sqrt{3}t)\right) + 3/7$$

y_h: $y'' - 4y' + 7y = 0$

$r^2 - 4r + 7 = 0$

$r = \frac{4 \pm \sqrt{16-28}}{2}$

$= 2 \pm \sqrt{-12}/2$

$= 2 \pm \cancel{\sqrt{4}} \cdot \sqrt{3} \cdot \sqrt{-1}/\cancel{2}$

$= 2 \pm \sqrt{3} \cdot \sqrt{-1}$

$y_h = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t)$

$= e^{2t} \cdot (C_1 \cos(\sqrt{3}t) + C_2 \sin(\sqrt{3}t))$

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y_p: $y'' - 4y' + 7y = 3$

Guess: $y_p = A$ (const)

$y = A$
$y' = 0$
$y'' = 0$

$0 - 4 \cdot 0 + 7 \cdot A = 3$

$7A = 3$

$A = 3/7$ $y_p = 3/7$

Ex: $y'' - 5y' + 6y = te^t$

$y = y_h + y_p = c_1 e^{2t} + c_2 e^{3t} + \left(\frac{1}{2}t + \frac{3}{4}\right)e^t$

y_h: $y'' - 5y' + 6y = 0$

$r^2 - 5r + 6 = 0$

$r_1 = 2$, $r_2 = 3$ $\longrightarrow$ $y_h = c_1 e^{2t} + c_2 \cdot e^{3t}$

y_p: $y'' - 5y' + 6y = te^t$

$f(t) = te^t$
$f'(t) = 1 \cdot e^t + t \cdot e^t$
$= (t+1)e^t$
$f''(t) = 1 \cdot e^t + (t+1)e^t$
$= (t+2)e^t$

Guess: $\boxed{y_p = (At+B)e^t}$

A, B are constants

$y = (At+B)e^t$

$y' = A \cdot e^t + (At+B)e^t = (At + A + B)e^t$

$y'' = A \cdot e^t + (At + A + B)e^t = (At + 2A + B)e^t$

$(At + 2A + B)e^t - 5(At + A + B)e^t + 6(At + B)e^t = te^t$

$(At + 2A + B) - 5(At + A + B) + 6(At + B) = t$

$(A - 5A + 6A)t = 1 \cdot t$

$+ (2A + B - 5A - 5B + 6B) + 0$

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$$\underline{2A} \cdot t + (\underbrace{-3A+2B}) = \underline{1} \cdot t + \underbrace{0}$$

$2A = 1$ and $-3A + 2B = 0$

$A = \underline{1/2}$ $\quad -3 \cdot 1/2 + 2B = 0$

$B = \underline{\frac{3}{4}}$

$$y_p = (At+B)e^t$$
$$= \underline{\underline{\left(\frac{1}{2}t + \frac{3}{4}\right)e^t}}$$

<u>Summary of method for finding y_p :</u> $\quad y'' + ay' + by = f(t)$

1. Look at $f(t)$, and its derivatives f', f'' and try to find a general "type" of function <u>$y(t)$</u> that depends on some parameters (constants).
   – (f, f', f'' should be special cases)
   This $y_p = y(t)$ is a good <u>guess</u> for solution.

2. Compute $y(t), y'(t), y''(t)$ and substitute in the diff. eqn. See if there are some <u>values</u> of the parameters that give solutions.
   Constant values!

3. If it doesn't work, try to multiply your guess by t and try again.