LECTURE 10

GRA 6035

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MATHEMATICS

Plan:

1. Differential equations
2. First order differential equations
   - i) Separable
   - ii) Linear
   - iii) Exact (next lecture)

Reading:

[MET] 24.1 – 24.2, (24.4 – 24.6)

Note on Integration

Integration methods

$$\int t^n\,dt = \frac{t^{n+1}}{n+1} + C \qquad \int e^t\,dt = e^t + C \qquad \int \frac{1}{t}\,dt = \ln|t| + C$$

Methods:

- integration by parts $\quad \int u' \cdot v\,dt = uv - \int uv'\,dt$
- Substitution $\quad \int f(u)\cdot u'\,dt = \int f(u)\,du$
- partial fractions $\quad \int \frac{1}{t^2-4t+3}\,dt$

$$= \int \frac{A}{t-3} + \frac{B}{t-1}\,dt$$

1 Differential equations

Ex:

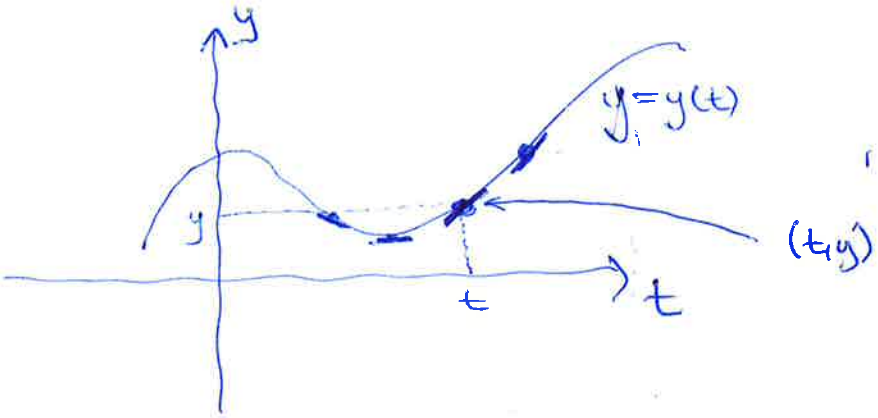
i) $y'(t) = 2t - 4$

ii) $y'(t) = 2y(t)$

iii) $y \cdot y' = 3y^2 \cdot t - \ln(ty)$

the function $y = y(t)$ is unknown

$y = y(t)$ is a solution to the diff. eqn. if it satisfies the requirement of the diff. eqn.



i) $y' = 2t - 4$

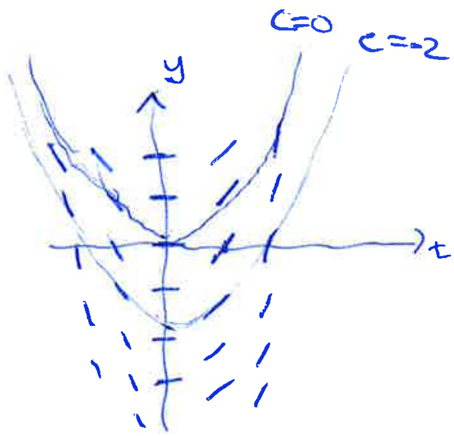
ii) $y' = 2y$

iii) $y' = \dfrac{3y^2t - \ln(ty)}{y}$

$$y' = F(y,t)$$

Ex: $y' = t$ $\quad F(y,t) = t$

$\underline{y = \frac{1}{2}t^2 + C}$ general solution



phase diagram

$y' = t$

$y = \int t\,dt = \underline{\frac{1}{2}t^2 + C}$

(2) First order differential equations

Order of a differential eqn = highest order of derivative in the diff. eqn.

First order: (diff. eqn.) An equation in y', y, t

Often it can be written as

$$\boxed{y' = F(t, y)}$$

In general:

i) The general solution of a first order diff. eqn. will depend on one undetermined coefficient C.

ii) A first order diff. eqn. with an initial condition has a unique solution called a particular solution.

Ex: $y' = 2t - 4$, $y(0) = 1$ ← initial condition $t=0, y=1$

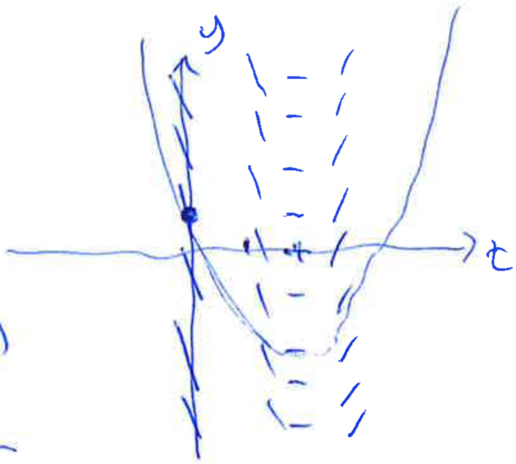
$y = \int 2t - 4 \, dt = t^2 - 4t + C$

$y = t^2 - 4t + C$ (general solution)

$y(0) = 1$: $1 = 0^2 - 4 \cdot 0 + C$

$C = 1$

$y = t^2 - 4t + 1$ (particular solution)



$\underbrace{\text{Differential eqn + initial condition}}_{\text{initial value problem}}$

Solve diff. eqn to find general solution, then use initial condition.

i) Separable diff eqn.

A first order diff. eqn. is called separable if it can be written in the form

$$\boxed{y' = f(y)\cdot g(t)}$$

Ex: $y' = yt = \underbrace{y}_{f(y)} \cdot \underbrace{t}_{g(t)}$ ← separable

$y' = y + t$ ← not separable

$y' = 2y = \underbrace{y}_{f(y)} \cdot \underbrace{2}_{g(t)}$ ← separable

$y' = 2t - 4 = \underbrace{1}_{f(y)} \cdot \underbrace{(2t-4)}_{g(t)}$ ← separable

Solution method for separable diff. eqn.

Ex: $y' = yt = y \cdot t \quad : y$

$\frac{1}{y} \cdot y' = t$

$\frac{1}{y} \cdot \frac{dy}{dt} = t$

$\frac{1}{y}\, dy = t\, dt$

$\int \frac{1}{y}\, dy = \int t\, dt$

$\ln|y| + C_1 = \frac{1}{2}t^2 + C_2$

$\ln|y| = \frac{1}{2}t^2 + C_2 - C_1$

$\ln|y| = \frac{1}{2}t^2 + K$ implicit solution

$|y| = e^{\frac{1}{2}t^2 + K}$

$y = \pm e^{\frac{1}{2}t^2 + K} = (\pm e^K) e^{\frac{1}{2}t^2}$

$y = Ae^{\frac{1}{2}t^2}$ general solution

Note: $\int \frac{1}{y}\, dy = \ln y + C$ since $(\ln y)'_y = \frac{1}{y}$

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but $\ln(y)$ is only defined for $y > 0$

When $y<0$: $(\ln(-y))'_y = \frac{1}{-y} \cdot (-1) = \frac{1}{y}$

Therefore:

$$\int \frac{1}{y}\, dy = \begin{cases} \ln(y) + C & , y>0 \\ \ln(-y) + C & , y<0 \end{cases} = \ln|y| + C$$

Note: $y' = y \cdot t \quad : y$

$\frac{1}{y} y' = t \longrightarrow \frac{1}{y}\frac{dy}{dt} = t$

$\frac{1}{y}\, dy = t\, dt$ separated form

$\boxed{\int \frac{1}{y}\, dy = \int t\, dt}$

$\downarrow$

$\int \frac{1}{y} \underbrace{y'\, dt}_{dy} = \int t\, dt$

$\boxed{\begin{array}{c} y = y(t) \\ dy = y'(t)\, dt \end{array}}$

$$\boxed{\int \frac{1}{y}\, dy = \int t\, dt}$$

General: Separable diff. eqn.

$$y' = f(y) \cdot g(t) \quad \Big| : f(y)$$

$$\frac{1}{f(y)} \cdot y' = g(t)$$

$$\int \frac{1}{f(y)} y' \, dt = \int g(t) \, dt$$

$$\boxed{\int \frac{1}{f(y)} \, dy = \int g(t) \, dt}$$

Ex: $y' = 2y$

$$\frac{1}{y} y' = 2$$

$$\int \frac{1}{y} \, dy = \int 2 \, dt$$

$$\ln|y| + C_1 = 2t + C_2 \longrightarrow \ln|y| = 2t + \underbrace{C_2 - C_1}_{c}$$

$$|y| = e^{2t+c} = e^{2t} \cdot e^{c}$$

$$y = \underbrace{\pm e^{c}}_{K} \, e^{2t} = K \cdot e^{2t}$$

i) Linear first order differential equations

A first order diff. eqn. is linear if it can be written

$$\boxed{y' + a(t)\cdot y = b(t)}$$

$$\Updownarrow$$

$$y' = b(t) - a(t)\cdot y \quad \text{(linear in } y\text{)}$$

Constant coefficients: $a(t)=a$ (and $b(t)=b$)

Homogeneous: $b(t)=0$

Ex: $y' + 3y = 2$ ← linear $\begin{cases} a(t)=3 \\ b(t)=2 \end{cases}$

Constant. coeff.

Method I: Integrating factor

Ex: $y' + 3y = 2 \quad | \cdot u(t) = \text{integrating factor}$

$$\underbrace{u\cdot y' + 3u\cdot y}_{} = 2u$$

Want: $u\cdot y' + u'\cdot y = 2u$ ← Need: $u' = 3u$, $u = e^{3t}$

$$(u\cdot y)' = 2u$$

$$u\cdot y = \int 2u\,dt$$

$$y = \frac{1}{u}\cdot\int 2u\,dt$$

$$y = \frac{1}{e^{3t}}\left(\int 2e^{3t}\,dt\right)$$

$$y = \frac{2}{e^{3t}}\left(\frac{1}{3}e^{3t} + C\right)$$

$$= 2/3 + 2Ce^{-3t} = \frac{2}{3} + Ke^{-3t}$$

It is possible to find the integrating factor by solving the sep. diff. eqn.

$$\boxed{u' = a(t)\cdot u}$$

The solution is

$$\boxed{u = e^{\int a(t)\,dt}}$$

General formula for the integrating factor

In general: Method of integrating factor

$$y' + a(t)\cdot y = b(t) \quad \Big|\cdot u(t) = e^{\int a(t)dt}$$

(for one integral)

$$\underbrace{u\cdot y' + a(t)\cdot u\cdot y}_{} = b(t)\cdot u$$

$$(u\cdot y)' = b(t)\cdot u$$

$$uy = \int b(t)\, u(t)\, dt$$

$$\boxed{y = \frac{1}{u(t)}\int b(t)\cdot u(t)\, dt}$$

Ex: Non-constant coefficients

$$y' + ty = 3t \quad \Big|\cdot e^{\frac{1}{2}t^2}$$

$$\left(y\cdot e^{\frac{1}{2}t^2}\right)' = 3t\cdot e^{\frac{1}{2}t^2}$$

$$y\cdot e^{1/2 t^2} = \int 3t\, e^{t^2/2}\, dt$$

$$= \int 3e^v dv$$

$$= 3e^v + C$$

Substitution
$v = t^2/2$
$dv = t\cdot dt$

linear: $a(t) = t$
$b(t) = 3t$

Integrating factor:
$u = e^{\int a(t)dt}$

$\int a(t)dt = \int t\,dt$
$= \frac{1}{2}t^2 + C$

$$\boxed{u = e^{\frac{1}{2}t^2}}$$

$$y\cdot e^{t^2/2} = 3e^{t^2/2} + C \quad \Big|: e^{t^2/2}$$

$$y = \underline{\underline{3 + Ce^{-t^2/2}}}$$

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<u>Ex:</u> Constant coeff's.

$$y' + ay = b \quad \Big| \cdot e^{at} \qquad (a, b \text{ constants})$$

$$(y \cdot e^{at})' = b \cdot e^{at}$$

$$y \cdot e^{at} = \int b\, e^{at}\, dt = \int b\, e^{v} \cdot \frac{dv}{a}$$

$$\boxed{\begin{array}{l} v = at \\ dv = a \cdot dt \end{array}}$$

$$= \frac{b}{a} \cdot (e^{v} + C)$$

$$y \cdot e^{at} = \frac{b}{a}(e^{at} + C) \quad \Big| : e^{at}$$

$$\underline{\underline{y = \frac{b}{a} + K e^{-at}}}$$

<u>Ex:</u> i) $y' + ty = 3t \Rightarrow y = \underline{3 + C \cdot e^{-t^2/2}}$, $e^{-t^2/2} \to 0$

ii) $y' + ay = b \qquad y = \frac{b}{a} + C \cdot e^{-at}$

i) $y(t) \to 3$ when $t \to \infty$

ii) $y(t) \to b/a$ when $t \to \infty$ if $a > 0$

$y(t) \to \pm\infty$ — " — if $a < 0$

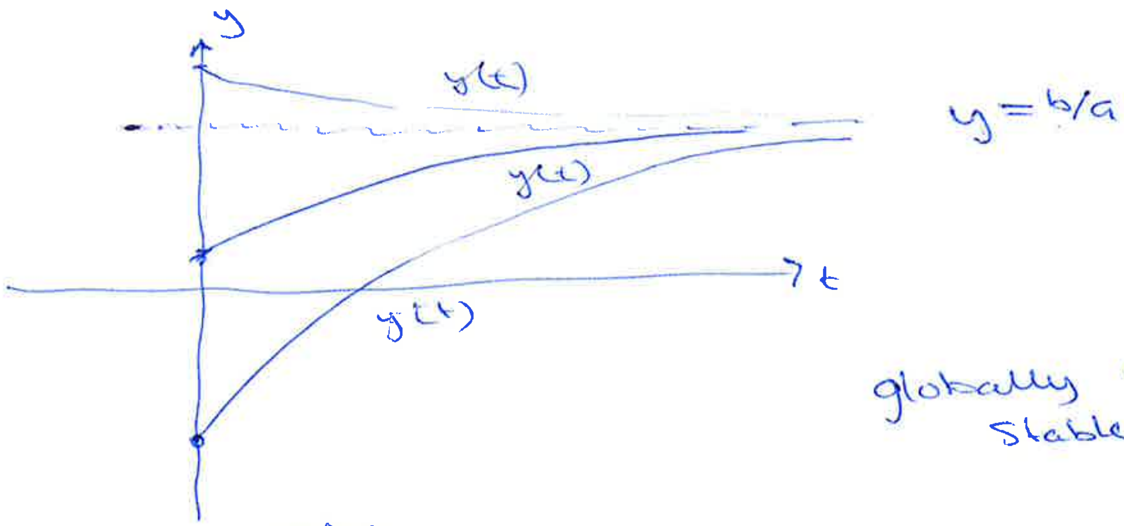
(If $a = 0$, then $y' + ay = b$ becomes $y' = b$, with solution $y = bt + C \to \pm\infty$ when $t \to \infty$)

Stability of linear differential equations

The general solution $y(t)$ is called <u>globally asymptotically stable</u> if $\lim_{t\to\infty} y(t) = \bar{y}$ exist and is independent of C.

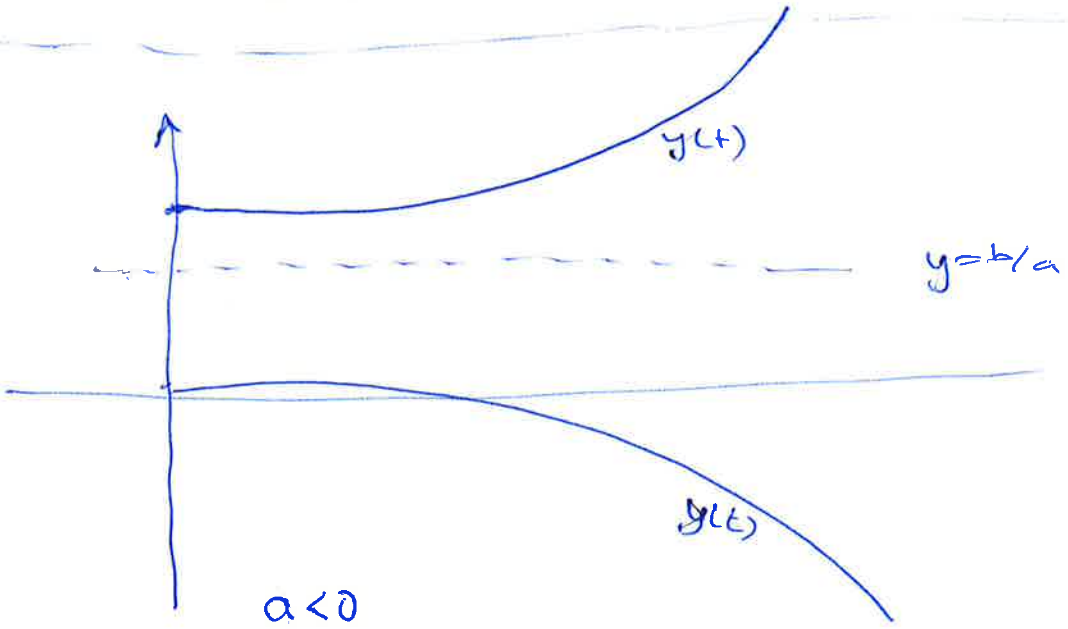
It is called <u>unstable</u> if $\lim_{t\to\infty} y(t)$ does not exist.

$$\boxed{y' + ay = b \quad \Rightarrow \quad y(t) = \frac{b}{a} + C\cdot e^{-at}}$$



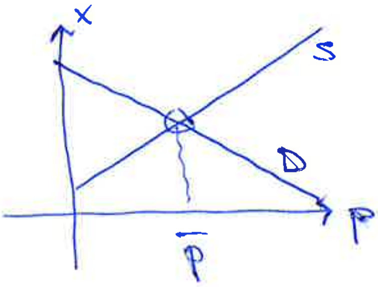
globally asympt. stable

$a > 0$



$a < 0$

Ex: $D = a - bp$ (demand)
$S = \alpha + \beta p$ (supply)
$a, b, \alpha, \beta > 0$ constants

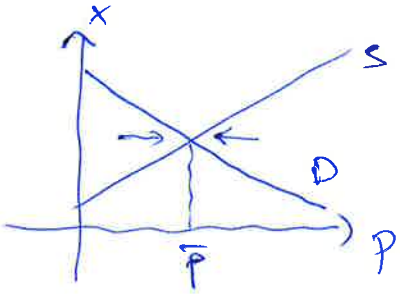


$p' = \lambda \cdot (D - S)$, $\lambda > 0$ constant

(differential equation modelling how quickly and in which direction p will change

$D > S$: $p' > 0$ $\rightarrow$ (p increases)

$D < S$: $p' < 0$ $\leftarrow$ (p decreases)



$a - b\bar{p} = \alpha + \beta\bar{p}$

$a - \alpha = (b + \beta)\bar{p}$

$$\bar{p} = \frac{a - \alpha}{b + \beta}$$ equilibrium price

Solution:

$$p' = \lambda \cdot (D - S) = \lambda \cdot ((a - bp) - (\alpha + \beta p))$$
$$= \lambda \cdot ((a - \alpha) - (b + \beta)p)$$

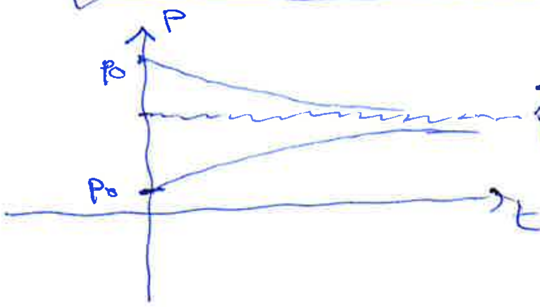
$$p' + \underbrace{\lambda \cdot (b + \beta)}_{\text{pos. const.}} p = \underbrace{\lambda (a - \alpha)}_{\text{pos. const.}}$$

$$y' + ay = b$$
$$\Downarrow$$
$$y = \frac{b}{a} + C \cdot e^{-at}$$

$$p = \frac{\lambda (a - \alpha)}{\lambda (b + \beta)} + C \cdot e^{-\lambda (b + \beta) t}$$

$$\boxed{p = \frac{a - \alpha}{b + \beta} + C \cdot e^{-\lambda (b + \beta) t}}$$

$\bar{p} = \frac{a - \alpha}{b + \beta}$ eq. price with globally asympt. stable price



$\bar{p} = \frac{a - \alpha}{b + \beta}$

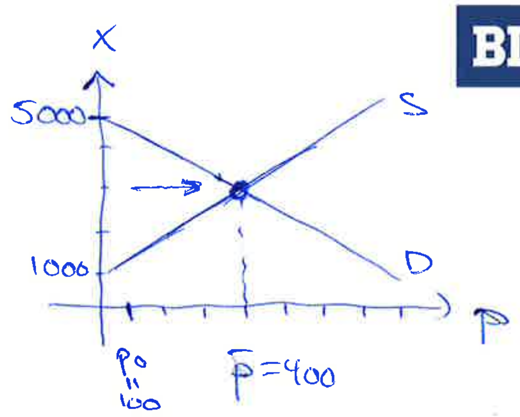
Solution gives precisely how the price will develope over time and approach $\bar{p}$.

Numerical example:

$$D = 5000 - 4p \qquad \bar{p} = \frac{4000}{10} = \underline{400}$$

$$S = 1000 + 6p$$

$$p' = \frac{1}{2}(D - S)$$



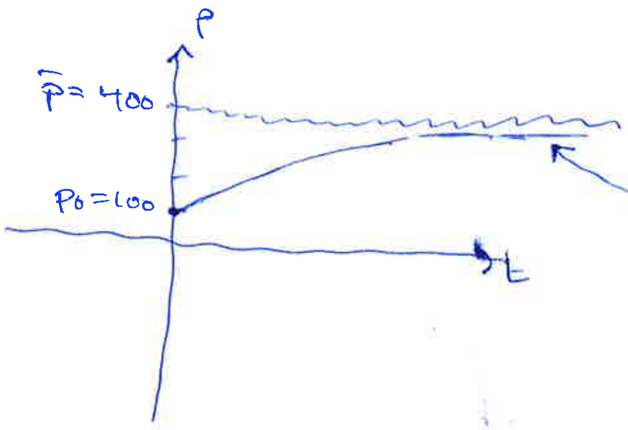
$$\underline{p(t) = 400 + Ce^{-5t}}$$

$$\underline{p(0) = 100}$$

$$400 + C \cdot e^{-5 \cdot 0} = 100$$

$$400 + C = 100$$

$$C = -300$$



$$p(t) = 400 - 300e^{-5t}$$

Postponed for next lecture: (in two weeks)

iii) Exact differential equations