LECTURE I

GRA 6035

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Plan:

1. Introduction to GRA 6035
2. Linear systems
3. Gaussian elimination
4. Rank

Reading:

[Syllabus]

[ME] 6.1, (6.2), 7.1-7.4, (7.5)

[LSGE] 1-3

[FORK 1003]

2 Linear systems

Ex:
$$\left.\begin{aligned} x+y+z&=3 \\ x+2y+4z&=7 \\ x+3y+9z&=11 \end{aligned}\right\}$$

A **linear equation** in the variables $x_1, x_2, \ldots, x_n$ has the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

where $a_1, a_2, \ldots, a_n$, b are given numbers.

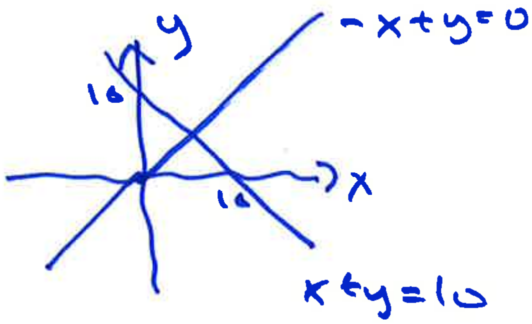
Ex: Not linear equations

$x^2+y^2=1 \qquad e^x+y=-1 \qquad xy=1$

Fact: The graph of a linear equation is

- a straight line if $n=2$
- a plane if $n=3$

Ex:
$$\begin{aligned} x+y&=10 \qquad & y&=10-x \\ -x+y&=0 \qquad & y&=x \end{aligned}$$



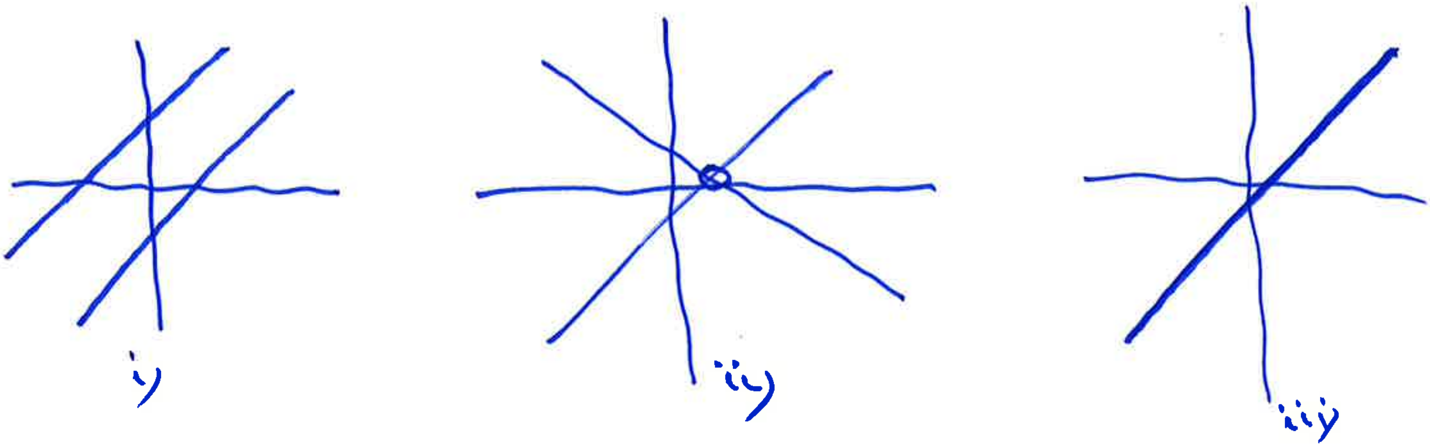
Basic fact:

A linear system has either

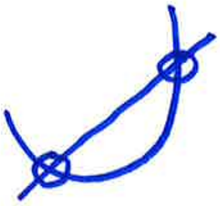
i) no solutions (inconsistent)

ii) one unique solution

iii) infinitely many solutions } (consistent)



(For non-lin. equations:



)

Definitions:

A linear system with $\underline{m}$ linear equations in $\underline{n}$ variables $x_1, \dots, x_n$ has the form

$$\boxed{\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}}$$

where a_{ij}, b_i are given numbers. This is called an <u>$m \times n$ - linear system</u>

A <u>solution</u> of the linear system is an n-tuple $(s_1, s_2, \dots, s_n)$ such that $x_1 = s_1$, $x_2 = s_2$, ..., $x_n = s_n$ satisfies all linear equations in the system.

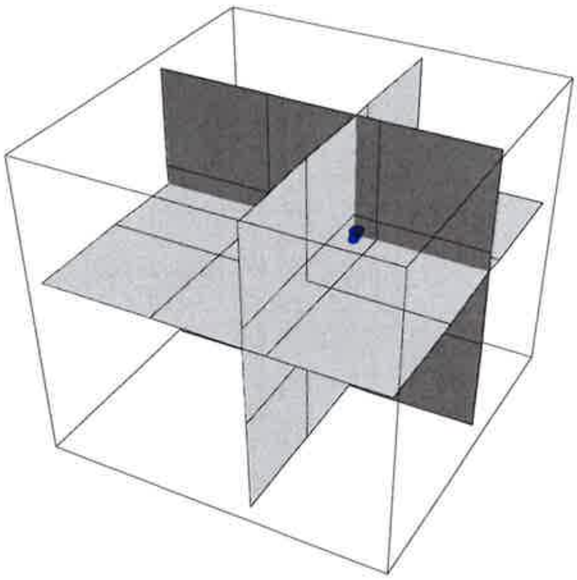
<u>Ex:</u>

$$\begin{aligned} x + y + z &= 3 \\ x + 2y + 4z &= 7 \\ x + 3y + 9z &= 13 \end{aligned}$$

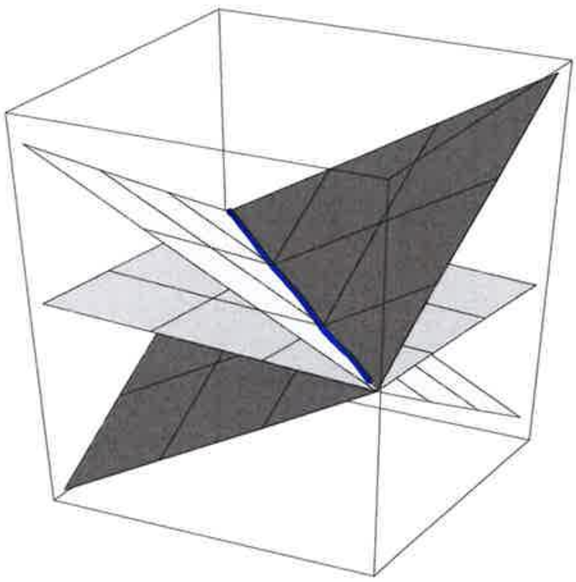
has solution <u>$(x,y,z) = (1,1,1)$</u>

EXAMPLE: Three equations in three variables. Each equation determines a plane in 3-space.

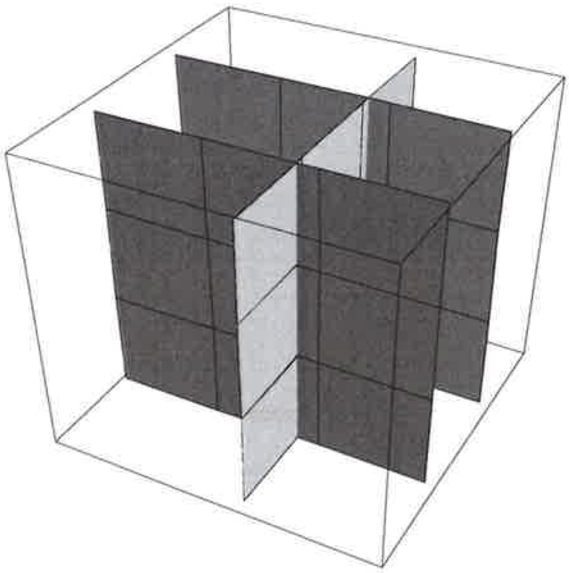
i) The planes intersect in one point. *(one solution)*



ii) The planes intersect in one line. *(infinitely many solutions)*



iii) There is not point in common to all three planes. *(no solution)*



Result Ⓐ:

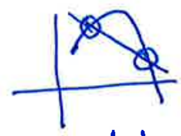
Any $m \times n$ linear system has either

i) one unique solution (consistent)

ii) no solutions (inconsistent)

iii) infinitely many solutions (consistent)

Non-linear case:



two solutions possible

Proof of A:

If there are two different solutions

$$(x_1, x_2, \dots, x_n) \neq (x_1', x_2', \dots, x_n')$$

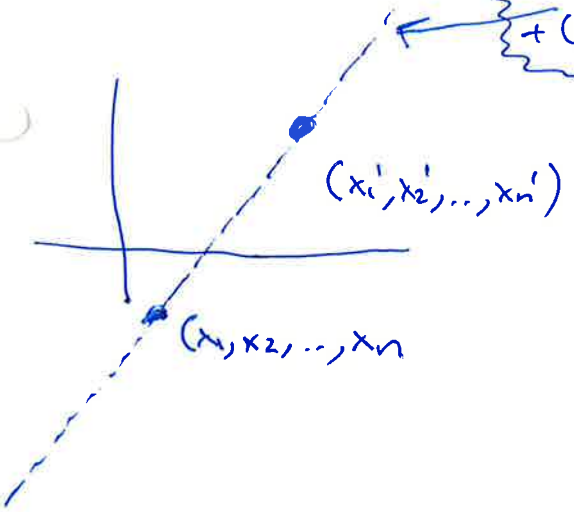
then

$$r(x_1, x_2, \dots, x_n) + (1-r)\cdot(x_1', x_2', \dots, x_n')$$
$$= (r x_1 + (1-r)x_1',\ r x_2 + (1-r)x_2',\ \dots,\ r x_n + (1-r)x_n')$$

is a solution for any number r. There are therefore infinitely many solutions

line through the two solutions.

$r(x_1, \dots, x_n) + (1-r)\cdot(x_1', \dots, x_n')$



This can be checked directly. Equation #i is satisfied since

$$a_{i1}\cdot(r x_1 + (1-r)x_1') + a_{i2}(r x_2 + (1-r)x_2') + \dots + a_{in}\cdot(r x_n + (1-r)x_n')$$
$$= r\cdot(a_{i1}x_1 + \dots + a_{in}x_n) + (1-r)\cdot(a_{i1}x_1' + \dots + a_{in}x_n')$$
$$= r\cdot b_i + (1-r)b_i = b_i$$

since $(x_1, \dots, x_n)$ and $(x_1', \dots, x_n')$ are solutions

So this equation holds for all i. Hence we have solution for all r. □

Solution techniques:

- substitution methods
- elimination methods

Ex:
$$x + y = 10 \quad (i)$$
$$-x + y = 0 \quad (ii)$$

Substitution:

$(i): \; y = 10 - x$

$\downarrow$ substitute y in (ii)

$(ii): \; -x + (10 - x) = 0$
$$-2x + 10 = 0$$
$$-2x = -10$$
$$\underline{x = 5}$$
$$\underline{y = 5}$$

Ex:
$$x + y + z = 3 \quad i)$$
$$x + 2y + 4z = 7 \quad ii)$$
$$x + 3y + 9z = 13 \quad iii)$$

i) $z = \underline{3 - x - y}$

$\downarrow$ substitute in ii) and iii)

ii) $x + 2y + 4(3 - x - y) = 7$
$$-3x - 2y = -5$$
iii) $x + 3y + 9(3 - x - y) = 13$
$$-8x - 6y = -14$$

$$y = \frac{-5 + 3x}{-2}$$
$$\underline{y = \frac{5}{2} - \frac{3}{2}x}$$

$$-8x - 6 \cdot \left(\frac{5}{2} - \frac{3}{2}x\right) = -14$$
$$-8x - 15 + 9x = -14$$

$$\underline{x = 1} \quad \underline{y = 1} \quad \underline{z = 1}$$

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Elimination methods

Ex:

$$\begin{array}{r} x+y=10 \\ -x+y=0 \\ \hline 0+2y=10 \end{array}$$

$\underline{y=5}$ $\underline{x=5}$

<u>Gaussian elimination</u> : - elimination method

(Gauss-Jordan elimination) - efficient, educational

③ Gaussian elimination:

Ex:

$$\begin{aligned} x+y+z &= 3 \\ x+2y+4z &= 7 \\ x+3y+9z &= 13 \end{aligned}$$

<u>coefficient matrix:</u>

$$A = \begin{array}{c} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ x & y & z \end{array} \end{array}$$

<u>augmented matrix</u>

$$\hat{A} = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 1 & 3 & 9 & 13 \end{array}\right)$$

Operations on linear systems that preserve the solutions

$$\begin{aligned} x + y &= 10 \\ -x + y &= 0 \end{aligned}$$

$$\hat{A} = \left(\begin{array}{cc|c} 1 & 1 & 10 \\ -1 & 1 & 0 \end{array}\right)$$

add the first row to the second row

$$\begin{aligned} x + y &= 10 \\ 2y &= 10 \end{aligned}$$

$$\left(\begin{array}{cc|c} 1 & 1 & 10 \\ 0 & 2 & 10 \end{array}\right)$$

In general, the following operations preserve solutions of linear system: **Elementary row operations**

i) Switch two rows

ii) Multiply a row by a constant $c \neq 0$

iii) Add c times one row to another row

Ex:

$$\begin{aligned} x+y+z &= 3 \\ x+2y+4z &= 7 \\ x+3y+9z &= 13 \end{aligned}$$

$\downarrow$ Augmented matrix

Add $(-1)\cdot R1$ to $R2$.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 1 & 3 & 9 & 13 \end{array}\right) \; -1$$

$\downarrow$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 1 & 3 & 9 & 13 \end{array}\right) \; -1$$

Eliminate variables in a systematic way using row operations

$\downarrow$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 8 & 10 \end{array}\right) \; -2$$

$\downarrow$

Echelon form.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right) \leadsto$$

$$\begin{aligned} x+y+z &= 3 \\ y+3z &= 4 \\ 2z &= 2 \end{aligned}$$

$$\begin{aligned} z &= 1 \\ y &= 1 \\ x &= 1 \end{aligned}$$

back substitution

A **pivot** is the first non-zero number in a row.

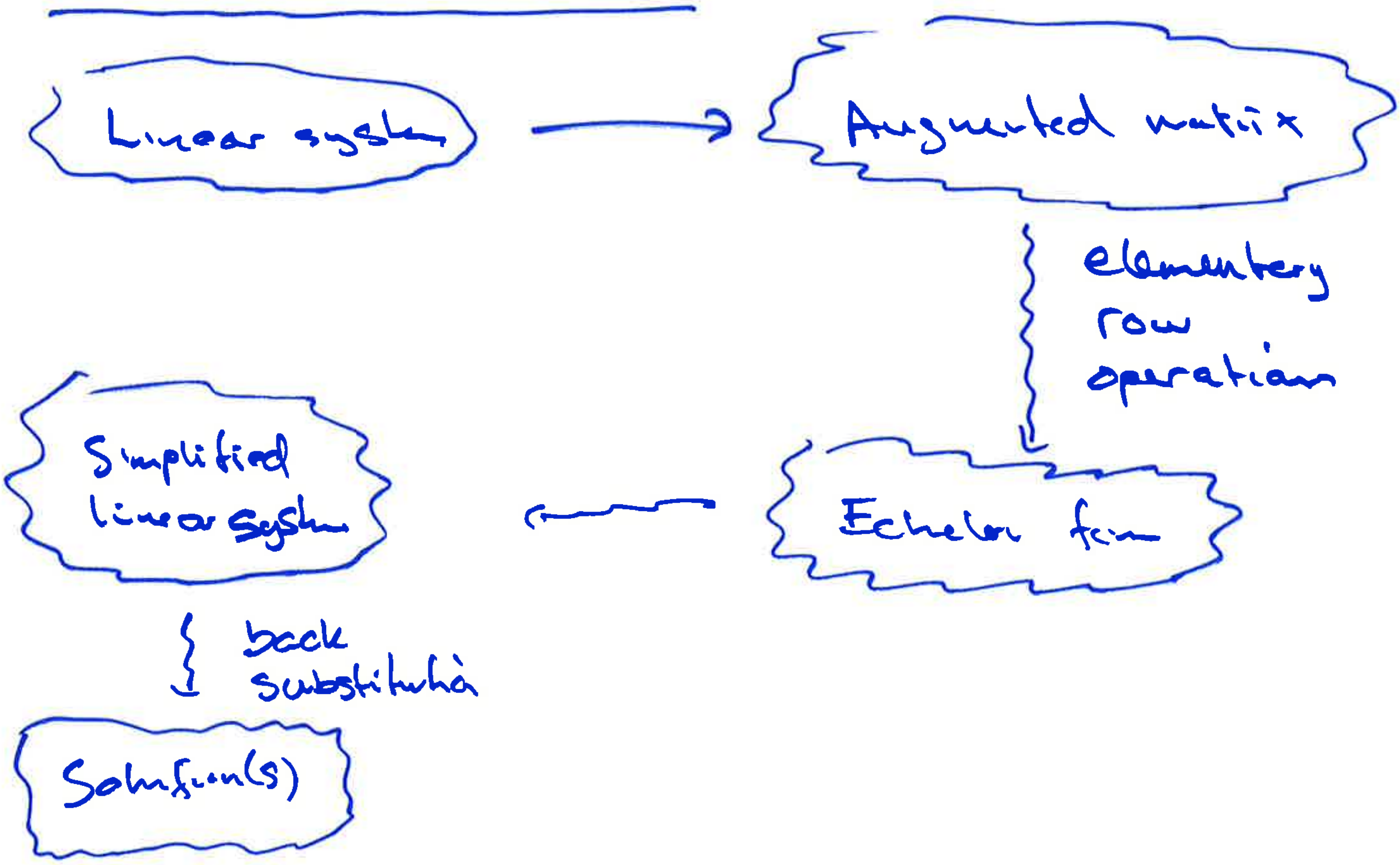
An **echelon form** is a matrix where

i) All rows consisting of zeros are at the bottom of the matrix

ii) All numbers under a pivot are zeros.

Fact: Starting from any matrix, you can always ~~transf~~ transform it into an **echelon form** using elementary row operations.

Gaussian elimination



However:

- the ~~reduced~~ echelon form is **not** unique, to have a unique form you have to find a **reduced** echelon form

Reduced echelon form:

Echelon form s.t.

i) All pivots are 1

ii) All numbers over a pivot are 0.

Ex:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right) \cdot \frac{1}{2} \longrightarrow \begin{aligned} x+y+z&=3 \\ y+3z&=4 \\ 2z&=2 \end{aligned}$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right) \begin{matrix} -1 \\ -3 \end{matrix}$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) -1$$

$$\downarrow$$

reduced echelon form

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \longrightarrow \begin{aligned} x&=1 \\ y&=1 \\ z&=1 \end{aligned}$$

* Any matrix can be transformed into an echelon form using elementary row operations (Gaussian elimination), but the echelon form is not unique.

* Any matrix can be transformed into a reduced echelon form using elementary row operations (Gauss-Jordan elimination), and then reduced echelon form is unique.

The pivot positions = positions where you have pivots in an echelon form. The pivot positions are unique.

Ex: $x+y+z=3$ → Echelon form: $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right)$

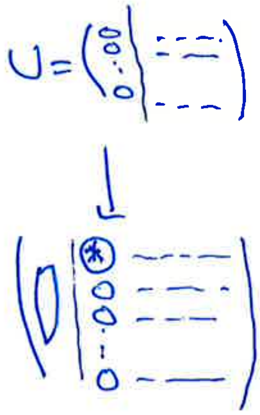
$\vdots$

Result B:

Any matrix can be reduced to an echelon form or to reduced echelon form using elementary row operations. The pivot positions and the reduced echelon form are unique, but the echelon form is not unique.

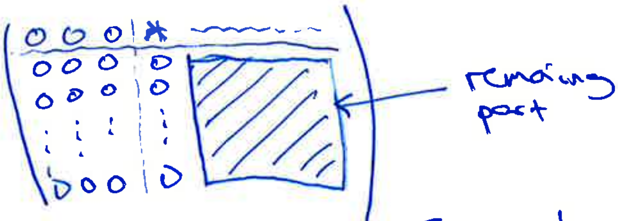
Proof of Result B and more details of Gauss/Gauss-Jordan elimination:

i) **Any matrix can be reduced to an echelon form using Elementary row operations.**



Start with any matrix U. Move to the right of any columns with only zeros, if any. Look at the first non-zero column, switch two rows if necessary to get a non-zero entry in the top corner. This is a pivot. Use it to get zeros under it.

Now look away from the first row, and look at the remaining part of the matrix.



Repeat the steps above. Since the new matrix is smaller than the original (one row less), we sooner or later get an echelon form this way.

Multiply each row with $\frac{1}{\text{pivot}}$ to set pivot $=1$. Use each pivot to set zeros over it, starting from the rightmost. You get a reduced echelon form.

ii) **If a matrix A can be reduced to reduced echelon forms U, V using elementary row operations, $U = V$.**

We have $(A \mid 0) \dashrightarrow (U \mid 0)$ and $(A \mid 0) \dashrightarrow (V \mid 0)$, and elementary row operations do not change solutions of linear systems. So $U \cdot \underline{x} = \underline{0}$ and $V \cdot \underline{x} = \underline{0}$ have the same solutions. Write $U = (c_1 | c_2 | \dots | c_n)$ and $V = (c_1' | c_2' | \dots | c_n')$ in terms of their columns. We have

$$c_i = x_1 c_1 + x_2 c_2 + \dots + x_{i-1} c_{i-1} \iff c_i \text{ non-pivot column in } U$$
$$\Updownarrow$$
$$c_i' = x_1 c_1' + x_2 c_2' + \dots + x_{i-1} c_{i-1}' \iff c_i' \text{ non-pivot column in } V$$

So U and V have the same pivot columns; They are in the positions, and the pivot columns are

$$\begin{matrix}1\\ \\ \\ \\ \\ \end{matrix}\begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix}, \begin{matrix} \\2\\ \\ \\ \\ \end{matrix}\begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix}, \begin{matrix} \\ \\3\\ \\ \\ \end{matrix}\begin{pmatrix}0\\0\\1\\\vdots\\0\end{pmatrix}, \dots, \begin{matrix} \\ \\ \\ \\r\\ \\ \\ \end{matrix}\begin{pmatrix}0\\0\\0\\\vdots\\1\\\vdots\\0\end{pmatrix}$$

To show that $U = V$, we must show that non-pivot columns are equal. But each non-pivot colum satisfy

$$\begin{cases} C_j = x_1 C_1 + \dots + \\ C_j' = x_1 C_1' + \dots + \end{cases}$$ (linear combination of pivot columns to the right)

and the pivot columns are equal, and the coeff's x_i are equal.
Hence $U = V$.

ii) You can always get from an echelon matrix to a reduced echelon matrix, using elementary row operations, without changing the pivot positions.

This follows from the last steps in i). $\square$

Ex:

$$x + y + z = 1$$
$$x + 2y + 3z = 2$$
$$2x + 3y + 4z = 4$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 3 & 4 & 4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 2 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

echelon form

$$x + y + z = 1$$
$$y + 2z = 1$$
$$0 = 1$$

inconsistent = no solution

Fact: inconsistent linear system (no solutions)

$$\Updownarrow$$

there is a pivot position in the last column (to the right of the line)

Ex:

$$x + y + z = 1$$
$$x + 2y + 3z = 2$$
$$2x + 3y + 4z = 3$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 3 & 4 & 3 \end{array}\right)$$

$$\downarrow$$

$$\begin{array}{cccc} x & y & z & \\ \end{array}$$
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

echelon form

Basic variables: columns with pivots : x, y

Free variables: columns without pivots : z

$$\begin{array}{ccc} x & y & z \\ \downarrow & \downarrow & \downarrow \end{array}$$

x, y: basic
z: free

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\underline{x} + y + z = 1$$
$$\underline{y} + 2z = 1$$

$$\boxed{\begin{aligned} x &= z \\ y &= 1 - 2z \end{aligned}}$$

$$x + (1-2z) + z = 1$$
$$x = z$$

Solutions: One free variable (z), one degree of freedom $\Rightarrow$ infinitely many solutions

$$\left.\begin{aligned} x &= z \\ y &= 1-2z \\ z &= \text{free} \end{aligned}\right\} \quad (x,y,z) = (z, 1-2z, z) \text{ where } z \text{ is free}$$

4 Rank

Defn: The rank of a matrix is the number of pivot positions in the matrix. Written $rk(A)$, $rk\,A$.

$$A = \begin{pmatrix} 1 & 2 & 7 & 4 \\ 0 & -1 & 2 & 5 \\ 1 & 1 & 9 & 9 \end{pmatrix} \begin{matrix} \\ \\ -1 \end{matrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 2 & 7 & 4 \\ 0 & -1 & 2 & 5 \\ 0 & -1 & +2 & 5 \end{pmatrix} \begin{matrix} \\ \\ -1 \end{matrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 2 & 7 & 4 \\ 0 & -1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow rk\,A = 2$$

echelon form

If A is an $m \times n$-matrix, then

$rk(A) \leq m$ and $rk(A) \leq n$

Rank and linear systems

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

$m \times n$ - linear system

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \qquad \hat{A} = \left(\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{array}\right)$$

coeff. matrix — augmented matrix

The number of solutions of the linear system is given by the pivot positions, and we can encode the essential information by computing:

$\operatorname{rk} A$, $\operatorname{rk} \hat{A}$

(rank of A) (rank of $\hat{A}$)

Result C:

For an $m \times n$ linear system with coefficient matrix A and augmented matrix $\hat{A}$, we have

$\operatorname{rk} A < \operatorname{rk} \hat{A} \iff$ no solutions (pivots in last col.)

$\operatorname{rk} A = \operatorname{rk} \hat{A} \iff$ at least one solution (no pivots in last col.)

Moreover, if $\operatorname{rk} A = \operatorname{rk} \hat{A}$, then there are

$n - \operatorname{rk}(A)$ degrees of freedom $\begin{cases} n - \operatorname{rk} A = 0: & \text{one unique sol.} \\ n - \operatorname{rk} A > 0: & \text{free var's, infinitely many sol's} \end{cases}$

A linear system is **homogeneous** if $b_1 = b_2 = \dots = b_m = 0$:

$$\left.\begin{matrix} a_{11} x_1 + \dots + a_{1n} x_n = 0 \\ \vdots \qquad\qquad \vdots \\ a_{m1} x_1 + \dots + a_{mn} x_n = 0 \end{matrix}\right\} \text{ homogeneous } m \times n \text{ linear system}$$

In this case, there is always at least one solution $x_1 = x_2 = \dots = x_n = 0$ (the *trivial* solution). That is, either this is the unique solution or there are inf. many sol's.

Ex: A 3×4 linear system that is homogeneous has *at least one free variable*:

$$\left(\begin{array}{cccc|c} \bullet & \bullet & \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet & \bullet & 0 \end{array}\right) \longleftarrow \text{maximum 3 pivots (one for each row) but depending on coeff's in matrix}$$

echelon form

$$\boxed{n - \operatorname{rk} A = 4 - \operatorname{rk} A \geq 1}$$