Lecture 9 (E)

GRA 6035 Mathematics

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Plan:

1. Envelope theorems
2. Bordered Hessians

Reading:

[ME] 19.2-19.3, (19.4-19.6)

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1 Envelope theorems

Ex: max $f(x) = -x^2 + 2ax + 4$ $\left\{\begin{array}{l} x: \text{variable} \\ a: \text{parameter} \end{array}\right.$

$\quad\;\; \| $

$\quad f(x;a)$

Question: For a given value of a, find the value of x that maximizes the function f.

Solve: $f'(x) = -2x + 2a = 0$

$\underline{x = a}$ ← Stationary pt.

$f''(x) = -2$ $\quad H(f) = (-2)$ neg. defn. for all x

$\Downarrow$

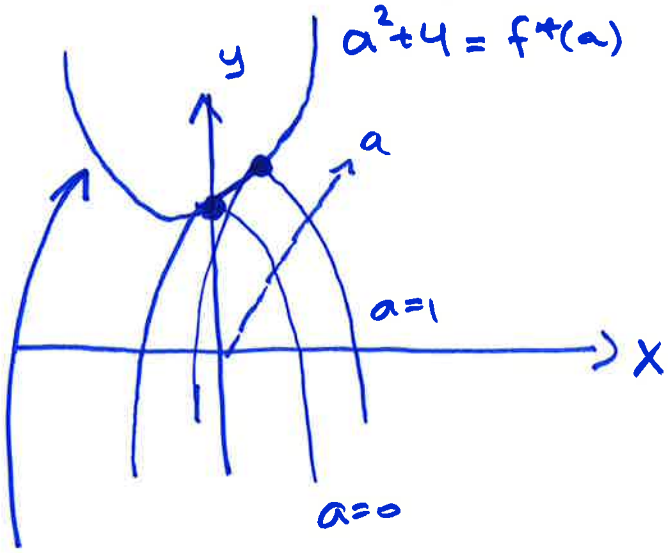
f concave

$\Leftarrow$ $x = a$ is global max

$x^*(a) = \underline{a}$: the maximum pt.

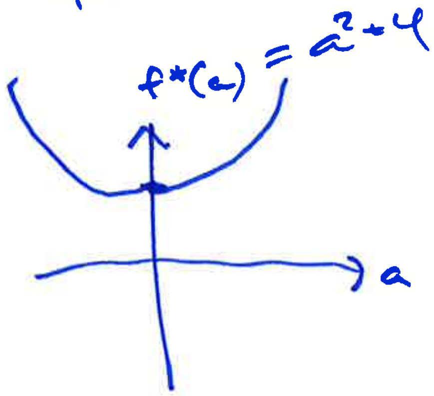
$f^*(a) = -a^2 + 2a \cdot a + 4 = \underline{a^2 + 4}$

$$\frac{df^*(a)}{da} = (a^2+4)'_a = \underline{2a}$$



$f(x) = -x^2+2ax+4$

Envelope



Envelope thm's

tells us how $f^*(a)$ changes with a, or more precisely, it gives a formula for

$$\frac{df^*(a)}{da}$$

Envelope thm (unconstrained case):

Optimization problem: max/min $f(\underline{x};a)$

(where $\underline{x} = x_1 \dots x_n$ are variables,
a is a parameter)

$x^*(a)$: max/min pt.

$f^*(a)$: max/min value
$= f(x^*(a))$

Env thm:

$$\boxed{\frac{df^*(a)}{da} = \frac{\partial f}{\partial a}(\underline{x}^*(a);a)}$$

Ex: max $f(x;a) = -x^2 + 2ax + 4$

Computed before: $\begin{cases} x^*(a) = a \\ f^*(a) = a^2 + 4 \\ \frac{d}{da} f^*(a) = 2a \end{cases}$

Env. thm. $\frac{df^*(a)}{da} = \frac{\partial f}{\partial a}(x^*(a);a)$

$= 2x^*(a) = 2a$

have to compute $x^*(a) = a$

Ex: $\pi(x,y) = 13x + qy - (500 + 4x + 2y + 0.04x^2 - 0.01xy + 0.01y^2)$

$\pi(x,y;q)$

Problem: max $\pi(x,y)$

$\frac{d\pi^*(q)}{dq} = \frac{\partial \pi}{\partial q}(x^*(q), y^*(q); q)$

$= y^*(q)$

$\pi'_x = 13 - 4 - 0.08x + 0.01y = 0 \quad | \cdot 100$

$\pi'_y = q - 2 + 0.01x - 0.02y = 0 \quad | \cdot 100$

$900 = 8x - y$

$8 \quad (q-2) \cdot 100 = -x + 2y$

One unique solution:

$15y = 900 + (q-2) \cdot 800$

$x = \frac{100q + 1600}{15}$ $\quad y = \frac{800q - 700}{15}$

$$H(\pi) = \begin{pmatrix} -0.08 & 0.01 \\ 0.01 & -0.02 \end{pmatrix} = \frac{1}{100} \cdot \begin{pmatrix} -8 & 1 \\ 1 & -2 \end{pmatrix}$$

$$\left.\begin{array}{l} D_1 = -0.08 < 0 \\ D_2 = \left(\frac{1}{100}\right)^2 \cdot (16-1) = \frac{15}{10.000} > 0 \end{array}\right\}$$

neg. defn.

$\Downarrow$

π concave

St. pt. is global max.

$$x^*(q) = \frac{100q + 1600}{15}$$

$$y^*(q) = \frac{800q - 700}{15}$$

$$\boxed{\text{Concl: } \quad \frac{d\pi^*(q)}{dq} = \frac{800q - 700}{15}}$$

$$\frac{d\pi^*(q)}{dq} > 0 \quad \text{when} \quad q > 7/8$$

We assume that $q \geq 7/8$ (otherwise $y^* < 0$)

Should have been stated as a constr. problem:

max $\pi(x,y)$ when $x \geq 0$, $y \geq 0$

$$\pi^*(q) = \frac{800}{15} \cdot \frac{1}{2} q^2 - \frac{700}{15} q + C$$

Ex: (As constrained problem)

$$\max \ \pi(x,y;q) \quad \text{when} \quad \begin{cases} x \geq 0 \\ y \geq 0 \end{cases}$$

$$\pi(x,y;q) = -500 + 9x + (q-2)y - 0.04x^2 + 0.01xy - 0.01y^2$$

$$\mathcal{L} = \pi(x,y;q) + \lambda_1 x + \lambda_2 y$$

$$\left.\begin{array}{l} \mathcal{L}'_x = \pi'_x + \lambda_1 = 0 \\ \mathcal{L}'_y = \pi'_y + \lambda_2 = 0 \\ x \geq 0 \ , \ y \geq 0 \\ \lambda_1 \geq 0 \ , \ \lambda_2 \geq 0 \\ \lambda_1 x = 0 \quad \lambda_2 y = 0 \end{array}\right\}$$

a) $\lambda_1 = \lambda_2 = 0$: $\left.\begin{array}{l}\pi'_x = 0 \\ \pi'_y = 0\end{array}\right\}$ as unconstrained case

gives $x = \frac{1600 + 100q}{15}$ $\quad y = \frac{800q - 700}{15}$

$x \geq 0, y \geq 0 \Leftrightarrow q \geq 7/8$.

Conclusion: $q \geq 7/8$ gives solution

$x^*(q) = \frac{1600 + 100q}{15}$ $\quad y^*(q) = \frac{800q - 700}{15}$

$\pi^*(q) = \frac{80}{3}q^2 - \frac{140}{3}q + \frac{80}{3}$ for $q \geq 7/8$

$\pi^*(7/8) = \frac{25}{4}$, $\pi^*(q)$ increasing for $q \geq 7/8$

$q < 7/8$: no solution in a)

b) $\lambda_1 > 0, \lambda_2 > 0$: $x = y = 0$, $\lambda_1 = -9 \quad \lambda_2 = 2 - q$ — no solution since $\lambda_1 < 0$

c) $\lambda_1 = 0, \lambda_2 > 0$: $y = 0 \quad x = 900/8$

$\lambda_2 = 7/8 - q$

Conclusion: $q < 7/8$ gives solution

$x^*(q) = 900/8 \quad y^*(q) = 0 \quad f^*(q) = \frac{25}{4}$

$q \geq 7/8$: no solution in c)

d) $\lambda_1 > 0, \lambda_2 = 0$: $x = 0, \ y = 50q - 100$

$\lambda_1 = -8 - \frac{1}{2}q$

$y \geq 0$ and $\lambda_1 > 0$

$\Updownarrow$

$q \geq 2$ and $q < -16$: no solution

Conclusion:

$$x^*(q) = \begin{cases} 900/8 & , q < 7/8 \\ \frac{1600 + 100q}{15} & , q \geq 7/8 \end{cases}$$

$$y^*(q) = \begin{cases} 0 & , q < 7/8 \\ \frac{800q - 700}{15} & , q \geq 7/8 \end{cases}$$

$$f^*(q) = \begin{cases} 25/4 & , q < 7/8 \\ \frac{80}{3}q^2 - \frac{140}{3}q + \frac{80}{3} & , q \geq 7/8 \end{cases}$$

Envelope thm (constrained case)

Problem: (Lagrange) max/min $f(\underline{x}; b)$ when $\begin{cases} g_1(\underline{x}; b) = 0 \\ g_2(\underline{x}; b) = 0 \\ \vdots \\ g_m(\underline{x}; b) = 0 \end{cases}$

write constraints like

$x^2 + y^2 = 10$

as

$x^2 + y^2 - 10 = 0$

$\underline{x}^*(b) = (x_1^*(b), \dots, x_n^*(b))$

max/min pt.

$f^*(b) = f(\underline{x}^*(b))$

max/min value

Env. thm: $\dfrac{df^*(b)}{db} = \dfrac{\partial L}{\partial b}(\underline{x}^*(b); \lambda^*(b); b)$

where we assume $(x^*(b); \lambda^*(b))$ satisfy FOC+C.

this is what we want to know

this gives a method for computing the derivative

i) Rewrite all constraints to the form $g(x_1, \dots, x_n) = 0$

ii) The env. thm. works in the same way for Kuhn-Tucker problems.

Ex: max $x+3y$ when $x^2+y^2 \le 10$

We considered this problem before:

$(x=1,\ y=3,\ \lambda=1/2)$ $\quad f=10$

$\uparrow$

Satisfies FOC + C + CSC and is max

$\{ \quad a=3,\ b=1,\ c=10$

max $x+ay$ when $x^2+by^2 \le c$

We know: $x^*(3,1,10)=1 \qquad f^*(3,1,10)=10$

$y^*(3,1,10)=3$

$\lambda^*(3,1,10)=1/2$

Env. thm: max $\underbrace{\cancel{x}+ay}_{f}$ when $\underbrace{x^2+by^2-c}_{g} \le 0$

$$\mathcal{L} = x+ay-\lambda\cdot(x^2+by^2-c)$$

$$\frac{\partial f^*(a,b,c)}{\partial a} = \frac{\partial \mathcal{L}}{\partial a}\left(x^*(\ ),\lambda^*(\ )\right) = y^*(a,b,c)$$

$$\frac{\partial f^*(a,b,c)}{\partial b} = \frac{\partial \mathcal{L}}{\partial b}\left(x^*(a,b,c), y^*(a,b,c); \lambda^*(a,b,c)\right) = -\lambda^*\cdot(y^*)^2$$

$$\frac{\partial f^*(a,b,c)}{\partial c} = \frac{\partial \mathcal{L}}{\partial c}\left(x^*(a,b,c); y^*(a,b,c); \lambda^*(a,b,c)\right) = \lambda^*(a,b,c)$$

Concl: $\frac{\partial f^*(3,1,10)}{\partial a} = y^*(3,1,10) = 3 \qquad \frac{\partial f^*(3,1,10)}{\partial b} = -\frac{9}{2} = -4.5$

$\frac{\partial f^*(3,1,10)}{\partial c} = \lambda^*(3,1,10) = 1/2 = 0.5$

Concl: $f^*(3,1,10) = 10$

$$\frac{\partial f^*}{\partial a}(3,1,10) = 3$$

$$\frac{\partial f^*}{\partial b}(3,1,10) = -4.5$$

$$\frac{\partial f^*}{\partial c}(3,1,10) = 0.5$$

When (a,b,c) is close to $(3,1,10)$, then

$$\begin{aligned} f^*(a,b,c) &\simeq 10 + 3\cdot(a-3) - 4.5(b-1) + 0.5(c-10) \\ &= 10 + 3a - 9 - 4.5b + 4.5 + 0.5c - 5 \\ &= \underline{0.5 + 3a - 4.5b + 0.5c} \end{aligned}$$

$$\begin{aligned} f^*(3.2, 0.9, 11) &= 10 + 0.2\cdot 3 + (-0.1)\cdot(-4.5) + 1\cdot 0.5 \\ &= 10 + 0.6 + 0.45 + 0.5 \\ &= \underline{\underline{11.55}} \end{aligned}$$

② Bordered Hessians

Lagrange problems:

$$\max/\min\ f(\underline{x}) \quad \text{when} \quad \begin{cases} g_1(\underline{x}) = a_1 \\ g_2(\underline{x}) = a_2 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases}$$

$$f(\underline{x}) = f(x_1, \dots, x_n)$$

$n = \#$variables $\qquad m = \#$constraints

Assume that $(\underline{x}^*; \underline{\lambda}^*)$ solves FOC + C, so that it is candidate for max/min.

Bordered Hessian:

$$B = \begin{array}{c} \\ \\ n \end{array} \begin{array}{c} \begin{array}{cc} m & n \end{array} \\ \left(\begin{array}{c|c} 0 & J \\ \hline J^T & H(\mathcal{L}) \end{array}\right) \end{array} \begin{array}{c} \\ m \\ \\ \end{array}$$

$(m+n) \times (m+n)$ matrix

$$J = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial x_1} & \frac{\partial g_m}{\partial x_2} & \cdots & \frac{\partial g_m}{\partial x_n} \end{pmatrix}$$

$$H(\mathcal{L}) = \begin{pmatrix} \mathcal{L}''_{x_1x_1} & \mathcal{L}''_{x_1x_2} & \cdots & \mathcal{L}''_{x_1x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{L}''_{x_nx_1} & \mathcal{L}''_{x_nx_2} & \cdots & \mathcal{L}''_{x_nx_n} \end{pmatrix}$$

Ex: max $x+3y$ when $x^2+y^2=10$

$$\mathcal{L} = x+3y - \lambda\cdot(x^2+y^2)$$

$$H(\mathcal{L}) = \begin{pmatrix} -2\lambda & 0 \\ 0 & -2\lambda \end{pmatrix}$$

$$J = \begin{pmatrix} 2x & 2y \end{pmatrix}$$

$$B = \left(\begin{array}{c|cc} 0 & 2x & 2y \\ \hline 2x & -2\lambda & 0 \\ 2y & 0 & -2\lambda \end{array}\right)$$

$n+m=3$

3×3-matrix

Idea: We compute $B(\underline{x}^*;\lambda^*)$ and try to find out if $\underline{x}^*$ is local max/min.

Ex: $(1,3;1/2)$ is cand. for max

$$B(1,3;1/2) = \begin{pmatrix} 0 & 2 & 6 \\ 2 & -1 & 0 \\ 6 & 0 & -1 \end{pmatrix}$$

Result:

i) $n-m=1$: Compute $|B(\underline{x}^*;\lambda^*)|$ and consider its sign:

Same sign as $(-1)^n$: local max

— " — $(-1)^m$: local min

__Ex:__ max $x+3y$ when $x^2+y^2=10$

$$\left.\begin{array}{l} n=2 \\ m=1 \end{array}\right\} \quad B = \begin{pmatrix} 0 & 2x & 2y \\ 2x & -2\lambda & 0 \\ 2y & 0 & -2\lambda \end{pmatrix}$$

__Cand:__

$(x=1, y=3, \lambda=1/2)$ $\quad B(1,3;1/2) = \begin{pmatrix} 0 & 2 & 6 \\ 2 & -1 & 0 \\ 6 & 0 & -1 \end{pmatrix}$

$$\begin{vmatrix} 0 & 2 & 6 \\ 2 & -1 & 0 \\ 6 & 0 & -1 \end{vmatrix} = -2\cdot(-2-0)+6\cdot(0+6)$$

$$= 4+36=40>0$$

__local max__

$(-1)^n = (-1)^2 = +1$

$(-1)^m = (-1)^1 = -1$

ii) __n-m is arbitrary:__ $(n-m>1)$

Compute the last $n-m$ leading principal minors of $B(\underline{x}^*;\lambda^*)$.

Signs are alternating + last sign is the sign of $(-1)^n$ $\Big\}$ (x^*,λ^*) is __local max__

Signs are all the sign of $(-1)^m$ $\Big\}$ $(x^*;\lambda^*)$ is __local min__

Ex: max/min $x^2y^2z^2$ when $x^2+y^2+z^2=3$

$\left.\begin{array}{l} n=3 \\ m=1 \end{array}\right\} n-m=2 \implies$ Must compute D_3, D_4 for $B(x^*, \lambda^*)$

Criterion:

$D_3 > 0$, $D_4 < 0 \implies$ local max

$D_3 < 0$, $D_4 < 0 \implies$ local min

$$\mathcal{L} = x^2y^2z^2 - \lambda\cdot(x^2+y^2+z^2)$$

$$\left.\begin{array}{l} \mathcal{L}'_x = 2xy^2z^2 - \lambda\cdot 2x = 0 \\ \mathcal{L}'_y = 2y\cdot x^2z^2 - \lambda\cdot 2y = 0 \\ \mathcal{L}'_z = 2z\cdot x^2y^2 - \lambda\cdot 2z = 0 \\ x^2+y^2+z^2 = 3 \end{array}\right\}$$

$\underline{x=y=z=1, \lambda=1}$

$f=1$

is this local max/min

$$B = \left(\begin{array}{c|ccc} 0 & 2x & 2y & 2z \\ \hline 2x & & & \\ 2y & & \mathcal{L}'' & \\ 2z & & & \end{array}\right) \qquad \mathcal{L}'' = \begin{pmatrix} 2y^2z^2-2\lambda & 4xyz^2 & \cdots \\ \vdots & \ddots & \\ & & \ddots \end{pmatrix}$$

$$B(1,1,1;1) = \left(\begin{array}{c|ccc} 0 & 2 & 2 & 2 \\ \hline 2 & 0 & 4 & 4 \\ 2 & 4 & 0 & 4 \\ 2 & 4 & 4 & 0 \end{array}\right)$$

$$B(1,1,1;1) = \left(\begin{array}{ccc|c} 0 & 2 & 2 & 2 \\ 2 & 0 & 4 & 4 \\ 2 & 4 & 0 & 4 \\ \hline 2 & 4 & 4 & 0 \end{array}\right)$$

$$D_3 = -2(-8) + 2(8) = 32 > 0$$

$$D_4 = \begin{vmatrix} 0 & 2 & 2 & 2 \\ 2 & 0 & 4 & 4 \\ 2 & 4 & 0 & 4 \\ 2 & 4 & 4 & 0 \end{vmatrix} \begin{matrix} \\ \\ -1 \\ -1 \end{matrix} = \begin{vmatrix} 0 & 2 & 2 & 2 \\ 0 & -4 & 0 & 4 \\ 0 & 0 & -4 & 4 \\ 2 & 4 & 4 & 0 \end{vmatrix}$$

$$= 2 \cdot \left(2 \cdot (0+16) + 4 \cdot (8+8) \right)$$

$$= -2 \left(32 + 64 \right) = -2 \cdot 96 = \underline{-192 < 0}$$

<u>Concl:</u> $(x,y,z) = (1,1,1)$ is local max for the Lagrange problem

What if you have a Kuhn-Tucker problem?

If $(\underline{x}^*; \underline{\lambda}^*)$ satisfy FOC + C + CSC, consider

$$B(\underline{x}^*; \underline{\lambda}^*)$$

with the following changes:

- Replace $\mathcal{J}$ in B with the submatrix where you only include rows corresponding to constraints that are binding at $\underline{x}^*$.
- Replace m with m', the number of constraints that are binding at $\underline{x}^*$

$$\Downarrow$$

$B(\underline{x}^*; \underline{\lambda}^*)$ $(n+m') \times (n+m')$-matrix

Compute the last $n-m'$ leading principal minors etc.