LECTURE 9 B

GRA 6035
MATHEMATICS

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OCT 16, 2014

Plan:

1. Envelope theorems
2. Bordered Hessians

Reading: [ME] 19.2–19.3, (19.4–19.6)

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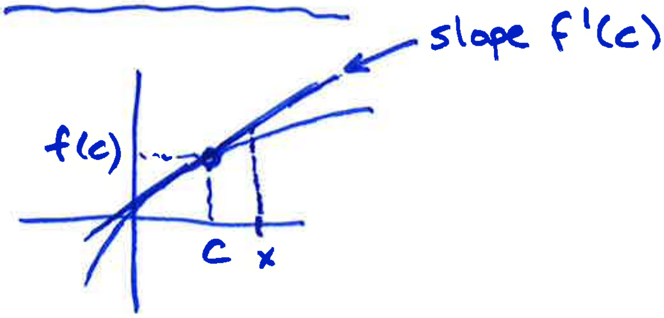
Linearization:



$$f(x) \simeq f(c) + (x-c)\cdot f'(c)$$
$$= f(c) + f'(c)\cdot(x-c)$$

for $x \simeq c$ (good approx. when x is close to c)

Linearization (one variable)

Version (several variables)

$$f(x_1,\dots,x_n) \simeq f(c_1,\dots,c_n) + f'_{x_1}(c_1\dots c_n)\cdot(x_1-c_1)$$
$$+ f'_{x_2}(c_1,\dots,c_n)\cdot(x_2-c_2)$$
$$+ \dots$$
$$+ f'_{x_n}(c_1,\dots,c_n)\cdot(x_n-c_n)$$

when $(x_1,\dots,x_n)$ is close to $(c_1,\dots,c_n)$.

Ex: $f(x,y) = \ln(x^2+y^2+1)$

Linearization at $(1,1)$: $f(1,1) = \ln(3)$

$f'_x = \dfrac{1}{x^2+y^2+1} \cdot 2x = \dfrac{2x}{x^2+y^2+1}$ $f'_x(1,1) = \dfrac{2}{3}$

$f'_y = \dfrac{2y}{x^2+y^2+1}$ $f'_y(1,1) = \dfrac{2}{3}$

Linearization : at $(1,1)$

$$f(x,y) = \ln(x^2+y^2+1) \simeq \ln(3) + \frac{2}{3}\cdot(x-1) + \frac{2}{3}(y-1)$$

$$= \underline{\frac{2}{3}x + \frac{2}{3}y + \left(\ln 3 - \frac{4}{3}\right)}$$

① Envelope theorems

Ex: $\max f(x;a) = -x^2+2ax+4$ $\begin{cases} x: \text{variable} \\ a: \text{parameter} \end{cases}$

Question: For a given value of a, find the value $x^*(a)$ that maximizes the function.

Solution:

$f'_x = -2x+2a = 0$

$\underline{x=a}$ ⟵ Stationary pt.

$f''_{xx} = -2$ $\quad H(f)(x) = (-2)$

neg. def. for all x

⇓

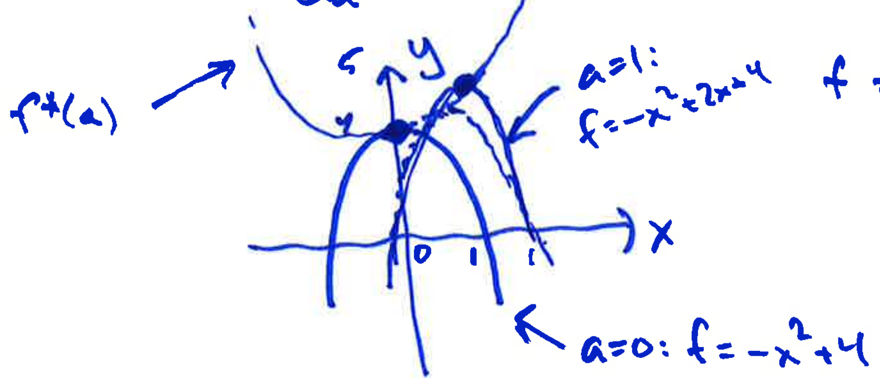
f concave (in x)

⇓

$x^*(a) = a$ is global max

$$f^*(a) = f(x^*(a)) = f(a)$$
$$= -a^2+2a\cdot a+4 = \underline{\underline{a^2+4}}$$

$$\frac{d}{da} f^*(a) = 2a$$



$f = -x^2+2a\cdot x+4$

Envelope thm:

Tell us how the maximum/minimum value $f^*(a)$ changes with a.

Envelope thm (unconstrained case):

Optimization problem: max/min $f(\underline{x}\,;a)$ $\left\{\begin{array}{l} x = x_1 \dots x_n \text{ variables} \\ a : \text{param.} \end{array}\right.$

$x^*(a)$: max/min pt.

$f^*(a)$: max/min value

Thm: $$\frac{df^*(a)}{da} = \frac{\partial f}{\partial a}(\underline{x}^*(a)\,;a)$$

this is what we want to compute

this gives us a method of computing it

Ex: $f(x;a) = -x^2 + 2xa + 4$

Env. thm: $$\frac{df^*(a)}{da} = \frac{\partial f}{\partial a}(x^*(a);a)$$

$$= 2x\Big|_{x=x^*(a)} = \underline{2a}$$

(Must still compute $x^*(a) = a$)

Ex: $\pi(x,y) = (13x + qy) - (500 + 4x + 2y + 0.04x^2 - 0.01xy + 0.01y^2)$

$\quad\shortparallel$

$\pi(x,y;q)$

max $\pi(x,y;q)$ for each given value of q

Env. thm:

$$\frac{d\pi^*(q)}{dq} = \left(\frac{\partial \pi}{\partial q}\right)(x^*(q), y^*(q), q)$$

$$= y^*(q)$$

$$\pi = 13x + qy - (500 + 4x + 2y + 0.04x^2 - 0.01xy + 0.01y^2)$$

$$\pi'_x = 13 - 4 - 0.08x + 0.01y = 0$$
$$\pi'_y = q - 2 + 0.01x - 0.02y = 0$$

$$\left.\begin{aligned} 9 &= 0.08x - 0.01y \\ q-2 &= -0.01x + 0.02y \end{aligned}\right| \begin{aligned} &\cdot 100 \\ &\cdot 100 \end{aligned}$$

$$\left.\begin{aligned} 8x - y &= 900 \\ -x + 2y &= (q-2)\cdot 100 \end{aligned}\right] 8\cdot$$

$$15y = 800q - 1600 + 900$$
$$= 800q - 700$$

$$y = \frac{800q - 700}{15} \qquad x = \frac{100q + 1600}{\ldots 15}$$

$$H(\pi) = \begin{pmatrix} -0.08 & 0.01 \\ 0.01 & -0.02 \end{pmatrix} \qquad \begin{aligned} D_1 &= -0.08 < 0 \\ D_2 &= 0.0016 - 0.0001 \\ &> 0 \end{aligned}$$

neg. def. for all (x,y)

$\Downarrow$

π is concave in (x,y)

$\Downarrow$

$$y^*(q) = \frac{800q - 700}{15} \qquad x^*(q) = \ldots$$

$$\boxed{\frac{d\pi^*(q)}{dq} = \frac{800q - 700}{15}} \qquad \boxed{\pi^*(q) = \pi(x^*(q), y^*(q))}$$

envelope thm.

note that $q \geq 7/8$ since $y \geq 0$

direct computation

Ex: (As constrained problem)

$$\max \ \pi(x,y;q) \quad \text{when} \quad \begin{cases} x \geq 0 \\ y \geq 0 \end{cases}$$

$$\pi(x,y;q) = -500 + 9x + (q-2)y - 0.04x^2 + 0.01xy - 0.01y^2$$

$$\mathcal{L} = \pi(x,y;q) + \lambda_1 x + \lambda_2 y$$

$$\left.\begin{array}{l} \mathcal{L}'_x = \pi'_x + \lambda_1 = 0 \\ \mathcal{L}'_y = \pi'_y + \lambda_2 = 0 \\ x \geq 0, \ y \geq 0 \\ \lambda_1 \geq 0, \ \lambda_2 \geq 0 \\ \lambda_1 x = 0 \quad \lambda_2 y = 0 \end{array}\right\}$$

a) $\lambda_1 = \lambda_2 = 0$: $\left.\begin{array}{l}\pi'_x = 0 \\ \pi'_y = 0\end{array}\right\}$ as unconstrained case

gives $x = \dfrac{1600 + 100q}{15}$ $\quad y = \dfrac{800q - 700}{15}$

$x \geq 0, y \geq 0 \Leftrightarrow q \geq 7/8$.

Conclusion: $q \geq 7/8$ gives solution

$$x^*(q) = \frac{1600+100q}{15} \qquad y^*(q) = \frac{800q - 700}{15}$$

$$\pi^*(q) = \frac{80}{3}q^2 - \frac{140}{3}q + \frac{80}{3} \quad \text{for } q \geq 7/8$$

$\pi^*(7/8) = \dfrac{25}{4}$, $\pi^*(q)$ increasing for $q \geq 7/8$

$q < 7/8$: no solution in a)

b) $\lambda_1 > 0, \lambda_2 > 0$: $\left.\begin{array}{l} x = y = 0 \\ \lambda_1 = -9 \quad \lambda_2 = 2 - q \end{array}\right\}$ no solution since $\lambda_1 < 0$

c) $\lambda_1 = 0, \lambda_2 > 0$: $y = 0$ $\quad x = 900/8$

$\lambda_2 = 7/8 - q$

Conclusion: $q < 7/8$ gives solution

$$x^*(q) = 900/8 \quad y^*(q) = 0 \quad f^*(q) = \frac{25}{4}$$

$q \geq 7/8$: no solution in c)

d) $\lambda_1 > 0, \lambda_2 = 0$: $x = 0$, $y = 50q - 100$

$\lambda_1 = -8 - \frac{1}{2}q$

$y \geq 0$ and $\lambda_1 > 0$

$\Updownarrow$

$q \geq 2$ and $q < -16$: no solution

Conclusion:

$$x^*(q) = \begin{cases} 900/8 & , q < 7/8 \\ \dfrac{1600+100q}{15} & , q \geq 7/8 \end{cases}$$

$$y^*(q) = \begin{cases} 0 & , q < 7/8 \\ \dfrac{800q - 700}{15} & , q \geq 7/8 \end{cases}$$

$$f^*(q) = \begin{cases} 25/4 & , q < 7/8 \\ \frac{80}{3}q^2 - \frac{140}{3}q + \frac{80}{3} & , q \geq 7/8 \end{cases}$$

Envelope thm (constrained case)

Problem: max/min $f(\underline{x}; b)$ subj. to $\begin{cases} g_1(\underline{x}; b) = 0 \\ \vdots \\ g_m(\underline{x}; b) = 0 \end{cases}$

$x^*(b)$: max/min pt.

$f^*(b)$: max/min value

Env. thm: $$\frac{df^*(b)}{db} = \frac{\partial L}{\partial b}\left(\underline{x}^*(b); \underline{\lambda}^*(b)\right)$$

this is what we want to compute

method for computing it

when $\left(\underline{x}^*(b), \underline{\lambda}^*(b)\right)$ solves FOC+C

Note: 1) When we use this result, we first rewrite all constraints to the form $g(\underline{x}) = 0$.

Ex: $x^2+y^2=10 \longrightarrow x^2+y^2-10=0$

2) The "same" applies to Kuhn-Tucker problems.

Ex: max $\overset{f(x,y)}{\overset{\shortparallel}{x + ay}}$ when $x^2 + by^2 \leq c$

$$\Updownarrow$$

$$x^2 + by^2 - c \leq 0$$

$$\mathcal{L} = (x + ay) - \lambda \cdot (x^2 + by^2 - c)$$

Env. thm:

$$\frac{\partial f^*(a,b,c)}{\partial a} = \frac{\partial \mathcal{L}}{\partial a}(x^*, y^*; \lambda^*) = \underline{y^*(a,b,c)}$$

$$\frac{\partial f^*(a,b,c)}{\partial b} = \frac{\partial \mathcal{L}}{\partial b}(x^*, y^*; \lambda^*) = -\lambda^* \cdot (y^*)^2$$

$$= \underline{-\lambda^*(a,b,c) \cdot (y^*(a,b,c))^2}$$

$$\frac{\partial f^*(a,b,c)}{\partial c} = \frac{\partial \mathcal{L}}{\partial c}(x^*, y^*; \lambda^*) = \underline{\lambda^*(a,b,c)}$$

FOC: $1 - \lambda \cdot 2x = 0$

$a - \lambda b \cdot 2y = 0$

C: $x^2 + by^2 \leq c$

CSC: $\lambda \geq 0$ and $\lambda \cdot \underset{0}{\underset{\shortparallel}{(x^2 + by^2 - c)}}$

Starting: $a = 3$, $b = 1$, $c = 10$

$1 - \lambda \cdot 2x = 0$ $\quad x^2 + y^2 \leq 10$

$3 - \lambda \cdot 2y = 0$ $\quad \lambda \geq 0$ and $\lambda(x^2 + y^2 - 10) = 0$

$\left.\begin{aligned} x &= 1/2\lambda \\ y &= 3/2\lambda \end{aligned}\right\}$ $x^2 + y^2 = 10$: $(1/2\lambda)^2 + (3/2\lambda)^2 = 10 \dashrightarrow \lambda = 1/2$

$\underline{x = 1,\ y = 3\ ;\ \lambda = 1/2}$

(check that this is actually the max)

$x^*(3,1,10) = 1$ $\quad \lambda^*(3,1,10) = \frac{1}{2}$

$y^*(3,1,10) = 3$ $\quad f^*(3,1,10) = 10$

Env. thm:

$\frac{\partial f^*}{\partial a} = \underline{3}$

$\frac{\partial f^*}{\partial b} = -\frac{1}{2} \cdot 3^2 = \underline{-4.5}$

$\partial f^*/\partial c = 1/2 = \underline{0.5}$

$$f^*(3,1,10) = 10$$

$$\frac{\partial f^*(3,1,10)}{\partial a} = 3 \qquad \frac{\partial f^*(a,b,c)}{\partial b} = -4.5 \qquad \frac{\partial f^*(a,b,c)}{\partial c} = 0.5$$

$$\begin{aligned} f^*(\overset{a}{\overset{\parallel}{3.2}}, \overset{b}{\overset{\parallel}{0.9}}, \overset{c}{\overset{\parallel}{11}}) &\simeq 10 + (3.2-3)\cdot 3 \\ &\quad + (0.9-1)\cdot(-4.5) \\ &\quad + (11-10)\cdot 0.5 \\ &= 10 + 0.6 + 0.45 + 0.5 \\ &= \underline{\underline{11.55}} \end{aligned}$$

(2) Bordered Hessians

Lagrange problem:

$$\max/\min\ f(\underline{x}) \quad \text{when} \quad \begin{cases} g_1(\underline{x}) = a_1 \\ g_2(\underline{x}) = a_2 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases}$$

$\underline{x} = (x_1, \dots, x_n)$

$n = \#\text{variables} \qquad m = \#\text{constraints}$

Assume that $(\underline{x}^*; \lambda^*)$ solves FOC + C.

We do not know if $\underline{x}^*$ is max/min.

__Bordered Hessian:__ $(m+n) \times (m+n)$-matrix

$$B = \begin{array}{cc} \left(\begin{array}{c|c} 0 & J \\ \hline J^T & \mathcal{L}'' = H(\mathcal{L}) \end{array}\right) & \begin{array}{c} m \\ n \end{array} \\ \begin{array}{cc} m & \qquad n \end{array} & \end{array}$$

where

$$J = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & & & \\ \frac{\partial g_m}{\partial x_1} & \frac{\partial g_m}{\partial x_2} & \cdots & \frac{\partial g_m}{\partial x_n} \end{pmatrix} \qquad H(\mathcal{L}) = \begin{pmatrix} \mathcal{L}''_{x_1x_1} & \mathcal{L}''_{x_1x_2} & \cdots & \mathcal{L}''_{x_1x_n} \\ \vdots & & & \\ \mathcal{L}''_{x_nx_1} & \mathcal{L}''_{x_nx_2} & \cdots & \mathcal{L}''_{x_nx_n} \end{pmatrix}$$

Hessian matrix of $\mathcal{L}$ w.r.t. $(x_1, \dots, x_n)$

Idea: We can decide if $(\underline{x}^*; \underline{\lambda}^*)$ is local max / min in the Lagrange problem by computing $B(\underline{x}^*; \underline{\lambda}^*)$.

Result:

1) $\underline{n - m = 1}$: $|B(\underline{x}^*; \lambda^*)|$ same sign as $(-1)^n$

$$\Rightarrow \underline{x}^* \text{ local max}$$

$|B(\underline{x}^*; \lambda^*)|$ same sign as $(-1)^m$

$$\Rightarrow \underline{x}^* \text{ local min}$$

Ex: max $x + 3y$ when $x^2 + y^2 = 10$ $\left\{ \begin{array}{l} n = 2 \\ m = 1 \end{array} \right.$

$$\mathcal{L} = x + 3y - \lambda(x^2 + y^2)$$

FOC: $\mathcal{L}'_x = 1 - \lambda \cdot 2x = 0 \quad \mathcal{L}'_y = 3 - \lambda \cdot 2y = 0$

C: $x^2 + y^2 = 10$

$$\downarrow$$
$$\vdots$$
$$\downarrow$$

$$(x, y; \lambda) = (1, 3; 1/2), \quad (-1, -3; -1/2)$$

$$\mathcal{L} = x + 3y - \lambda\cdot(x^2+y^2) \qquad H(\mathcal{L}) = \begin{pmatrix} -2\lambda & 0 \\ 0 & -2\lambda \end{pmatrix}$$

$$B = \left(\begin{array}{c|cc} 0 & 2x & 2y \\ \hline 2x & -2\lambda & 0 \\ 2y & 0 & -2\lambda \end{array}\right) \qquad B(1,3;\tfrac{1}{2}) = \left(\begin{array}{c|cc} 0 & 2 & 6 \\ 2 & -1 & 0 \\ 6 & 0 & -1 \end{array}\right)$$

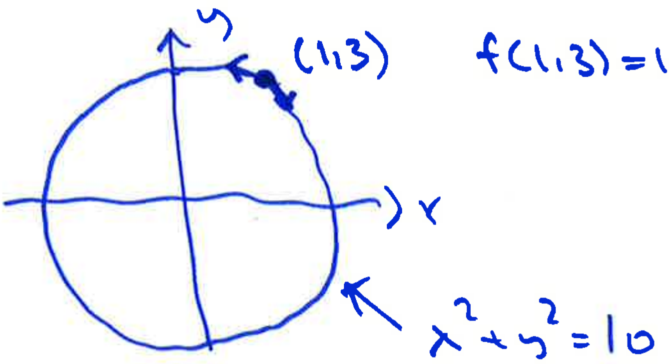
$$|B(1,3;\tfrac{1}{2})| = -2\cdot(-2-0) + 6\cdot(0+6)$$

$$= 4 + 36 = 40 > 0$$

Same sign as $(-1)^n = (-1)^2 = +1$

$$\Downarrow$$

$(x,y) = (1,3)$ is <u>a local max</u>



$f(1,3) = 10$

2) <u>n-m arbitrary:</u> $(n-m \geq 2)$

Compute $B(\underline{x}^*; \lambda^*)$ and its last $n-m$ leading principal minors.

- If the signs of these leading principal minors are alternating, and the sign of $|B(x^*;\lambda^*)| = D_{n+m}$ is equal to the sign of $(-1)^n$, then $\underline{x}^*$ is <u>local max</u>
- If the signs of these leading principal minors are the same as the sign of $(-1)^m$, then $\underline{x}^*$ is <u>local min</u>.

<u>Ex:</u> max/min $x^2y^2z^2$ when $x^2+y^2+z^2=3$

$n=3,\ m=1 \qquad n-m=2 \qquad n+m=4$

$\Downarrow$

B is 4×4, we must compute D_3, D_4

local max $\Leftarrow D_3>0,\ D_4<0 \qquad + \ -$

local min $\Leftarrow D_3<0,\ D_4<0 \qquad - \ -$

Ex: max/min $x^2y^2z^2$ when $x^2+y^2+z^2=3$

Lagrange problem: $n=3$, $m=1$

A solution to FOC+C is $\left\{ \begin{array}{l} x^*=1 \\ y^*=1 \\ z^*=1 \\ \lambda^*=1 \end{array} \right.$

$$\mathcal{L} = x^2y^2z^2 - \lambda\cdot(x^2+y^2+z^2)$$

$$B = \left(\begin{array}{c|ccc} 0 & 2x & 2y & 2z \\ \hline 2x & 2y^2z^2-2\lambda & 4xyz^2 & 4xzy^2 \\ 2y & 4xyz^2 & 2x^2z^2-2\lambda & 4x^2yz \\ 2z & 4xzy^2 & 4x^2yz & 2x^2y^2-2\lambda \end{array}\right)$$

$(-1)^n = -1$

$(-1)^m = -1$

$$B(1,1,1;1) = \left(\begin{array}{c|cc|c} 0 & 2 & 2 & 2 \\ \hline 2 & 0 & 4 & 4 \\ 2 & 4 & 0 & 4 \\ \hline 2 & 4 & 4 & 0 \end{array}\right)$$

Compute D_3, D_4: $\left\{ \begin{array}{l} \text{local max: } D_3>0,\ D_4<0 \\ \text{local min: } D_3<0,\ D_4<0 \end{array} \right.$

$\left. \begin{array}{l} D_3 = 32 \\ D_4 = -192 \end{array} \right\}$ $(x,y,z)=(1,1,1)$ is <u>local max</u>

What if you have a Kuhn-Tucker problem?

If $(\underline{x}^*; \lambda^*)$ satisfy FOC+C+CSC, consider

$$B(\underline{x}^*; \underline{\lambda}^*)$$

with the following changes:

- Replace J in B with the submatrix where you only include rows corresponding to constraints that are binding at $\underline{x}^*$.

- Replace m with m', the number of constraints that are binding at $\underline{x}^*$

$\Downarrow$

$B(\underline{x}^*; \underline{\lambda}^*)$ $(n+m') \times (n+m')$-matrix

Compute the last $n-m'$ leading principal minors etc.