LECTURE 8 (F)

GRA 6035 MATHEMATICS

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Plan:

1. Lagrange multipliers
2. Second order conditions (SOC)

Reading:

[ME] 19.1, 19.4, (18.1 – 18.7)

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(1) Lagrange multipliers

Ex: max $3x+4y$ when $x^2+y^2=25$

$\mathcal{L} = 3x+4y-\lambda\cdot(x^2+y^2)$ λ: Lagrange multiplier

FOC: $\mathcal{L}'_x = 3-\lambda\cdot 2x = 0$ $x = \frac{3}{2\lambda}$

$\mathcal{L}'_y = 4-\lambda\cdot 2y = 0$ $y = \frac{4}{2\lambda}$ $\Big\}\ \lambda \neq 0$

C: $x^2+y^2 = 25$

$\frac{9+16}{(2\lambda)^2} = 25$

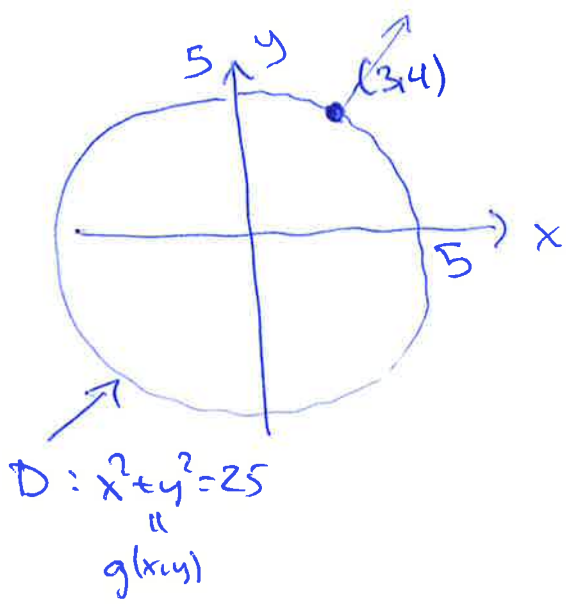
$\boxed{\lambda = 1/2: \ x=3,\ y=4;\ \lambda=1/2\ \ f=25}$ $(2\lambda)^2 = 1$ $2\lambda = \pm 1$

$\lambda=-1/2: \ x=-3,\ y=-4;\ \lambda=-1/2\ \ f=-25$ $\lambda = \pm 1/2$

Best candidate for max

In this problem, it turns out that $(x,y)=(3,4)$ is the max (D is bounded, no pts where NDCQ fails). What is the interpretation of $\lambda = 1/2$?

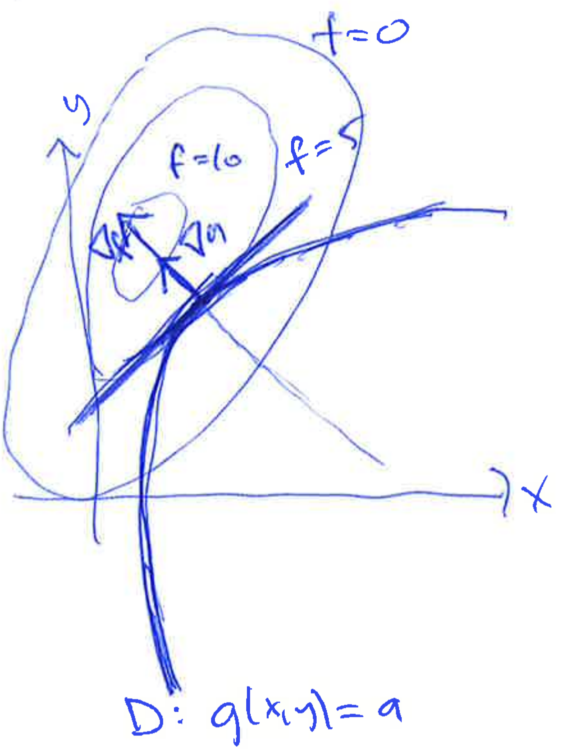
Gradient: $\nabla f = \begin{pmatrix} f'_{x_1} \\ f'_{x_2} \\ \vdots \\ f'_{x_n} \end{pmatrix}$ the gradient of f has the where f increases at the quickest rate.



$\nabla f = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

FOC: $\boxed{\nabla f = \lambda \cdot \nabla g}$

$$\begin{pmatrix} f'_{x_1} \\ f'_{x_2} \\ \vdots \\ f'_{x_n} \end{pmatrix} = \lambda \cdot \begin{pmatrix} g'_{x_1} \\ g'_{x_2} \\ \vdots \\ g'_{x_n} \end{pmatrix}$$



FOC: $\mathcal{L}'_{x_1} = f'_{x_1} - \lambda \cdot g'_{x_1} = 0$

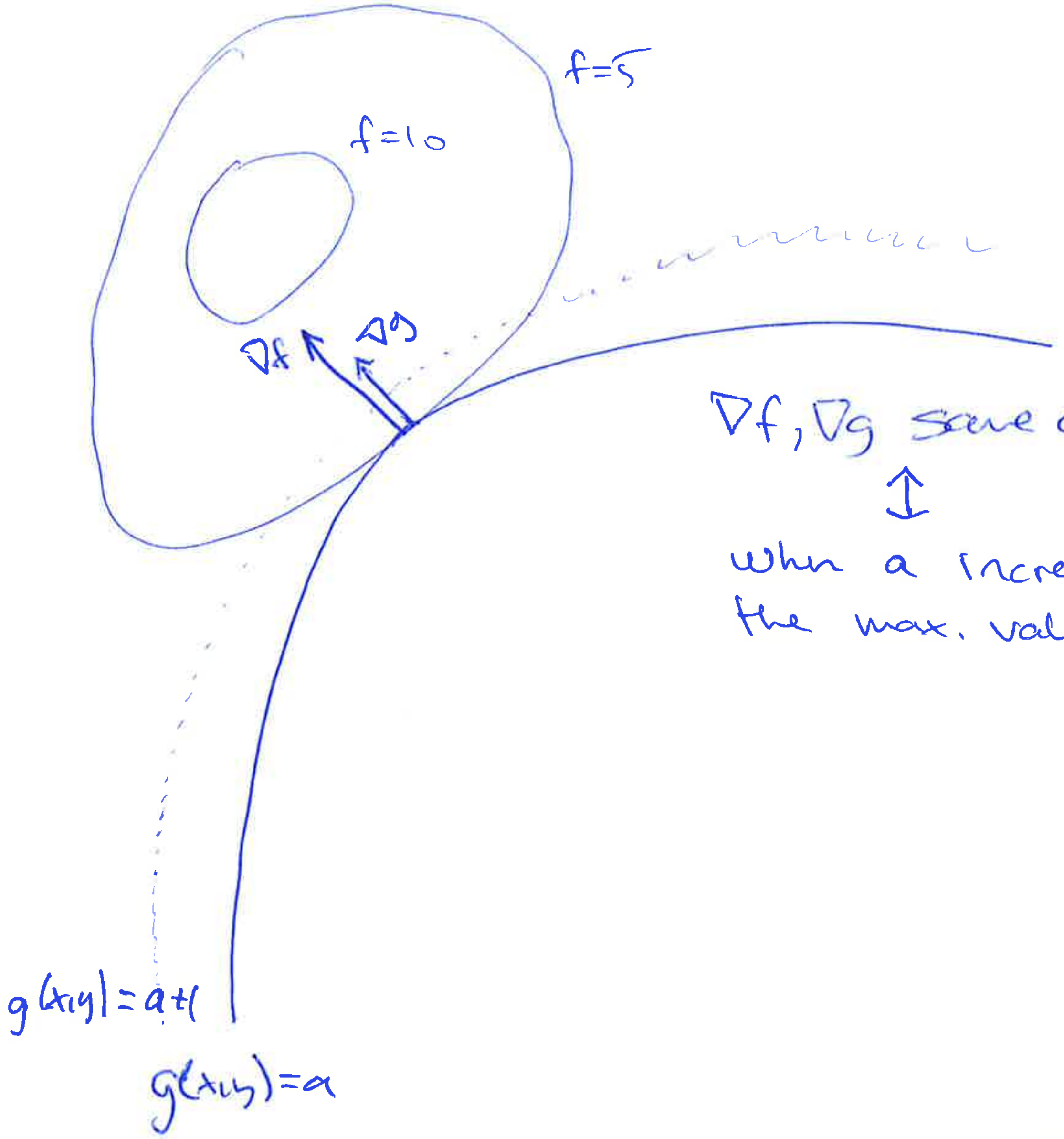
$\mathcal{L}'_{x_2} = f'_{x_2} - \lambda \cdot g'_{x_2} = 0$

$\vdots$

$\mathcal{L}'_{x_n} = f'_{x_n} - \lambda \cdot g'_{x_n} = 0$

Note: $\lambda > 0$ means that $\nabla f, \nabla g$ points in the same direct.

$\lambda < 0$ means that $\nabla f, \nabla g$ point in opposite direct.



$\nabla f, \nabla g$ same direction $\Leftrightarrow \lambda > 0$

$\Updownarrow$

when a increases, the max. value will increase

Result:

Let max $f(\underline{x})$ when $\begin{cases} g_1(\underline{x}) = a_1 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases}$ be a Lagrange problem, with solution $\underline{x}^*(a_1, a_2, \ldots, a_m)$; $\underline{\lambda}^*(a_1 \ldots, a_m)$ that satisfies FOC+C, and let

$$f^*(a_1 \ldots, a_m) = f(x^*(a_1 \ldots, a_m))$$

← optimal value function, maximum value for each $a_1 \ldots, a_m$.

Then $$\lambda_i^*(a_1 \ldots, a_m) = \frac{\partial f^*(a_1 \ldots, a_m)}{\partial a_i}$$

λ_i^* is approx. equal to the change in $f^*(a_1 \ldots, a_m)$ when a_i increases with 1

Look at the example:

$$\max \quad 3x+4y \quad \text{when} \quad x^2+y^2=25$$

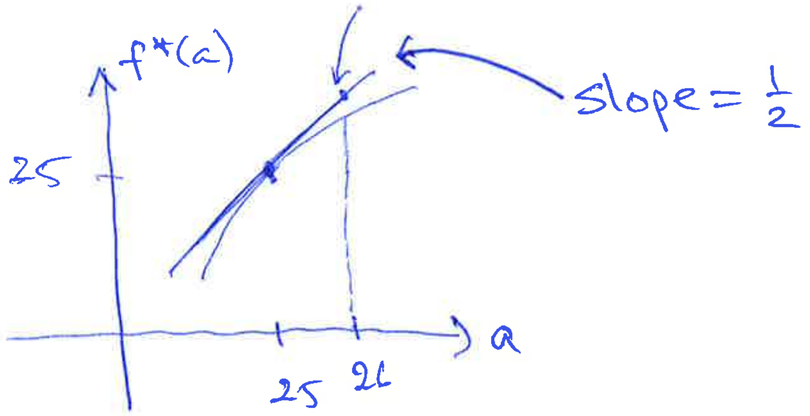
$$= \max \quad 3x+4y \quad \text{when} \quad x^2+y^2=a$$

$\lambda^*(25) = 1/2$

$\underline{a=25:}$ $\quad x^*(25)=3 \quad y^*(25)=4 \quad f^*(25)=25$

$\underline{a=26:}$ $\quad f^*(26) \cong f^*(25) + (26-25)\cdot\lambda^*(25)$

$$= 25 + 1\cdot\tfrac{1}{2} = 25{,}5$$



In case we consider a minimum problem, or a Kuhn-Tucker problem, the result is exactly the same.

Linear approximation of a function

Let $f(x_1, \dots, x_n)$ be a C^1 function in a neighbourhood at the point $(x_1^*, \dots, x_n^*)$. Then the linear approximation of f is

$$\boxed{\begin{aligned} f(x_1, \dots, x_n) \simeq f(x_1^*, \dots, x_n^*) &+ \frac{\partial f}{\partial x_1}(x_1^*, \dots, x_n^*) \cdot (x_1 - x_1^*) \\ &+ \frac{\partial f}{\partial x_2}(x_1^*, \dots, x_n^*) \cdot (x_2 - x_2^*) \\ &+ \dots + \frac{\partial f}{\partial x_n} \cdot (x_1^*, \dots, x_n^*) \cdot (x_n - x_n^*) \end{aligned}}$$

Ex: $f(x,y) = e^{xy}$ at $(0,0)$ and $(1,1)$

$f(0,0) = \underline{1}$

$f'_x = e^{xy} \cdot y \quad f'_x(0,0) = \underline{0}$

$f'_y = e^{xy} \cdot x \quad f'_y(0,0) = \underline{0}$

When (x,y) close to $(0,0)$:

$$f(x,y) \cong 1 + 0 \cdot (x-0) + 0 \cdot (y-0)$$
$$= 1$$

$f(1,1) = e \quad f'_x(1,1) = e \quad f'_y(1,1) = e$

When (x,y) close to $(1,1)$:

$$f(x,y) \simeq e + e \cdot (x-1) + e \cdot (y-1)$$
$$= ex + ey - e$$

Ex: $\min \underbrace{2x^2+y^2+3z^2}_{= f(x,y,z)}$ when $\begin{cases} x-y+2z \geq 3 \\ x+y \geq 3 \end{cases}$

$= \max -(2x^2+y^2+3z^2)$ when $\begin{cases} -(x-y+2z) \leq -3 \\ -(x+y) \leq -3 \end{cases}$

KT problem in std. form

$$\mathcal{L} = -2x^2 - y^2 - 3z^2 - \lambda_1 \cdot (-(x-y+2z)) - \lambda_2(-(x+y))$$

$$\mathcal{L} = -2x^2 - y^2 - 3z^2 + \lambda_1(x-y+2z) + \lambda_2(x+y)$$

FOC:

$$\mathcal{L}'_x = -4x + \lambda_1 + \lambda_2 = 0$$
$$\mathcal{L}'_y = -2y - \lambda_1 + \lambda_2 = 0$$
$$\mathcal{L}'_z = -6z + 2\lambda_1 = 0$$

C: $x-y+2z \geq 3$, $x+y \geq 3$

CSC: $\lambda_1 \geq 0$ and $\lambda_1 \cdot (x-y+2z-3) = 0$

$\lambda_2 \geq 0$ and $\lambda_2 \cdot (x+y-3) = 0$

Solve: FOC + C + CSC

FOC: $x = \dfrac{\lambda_1+\lambda_2}{4}$ $\quad y = \dfrac{\lambda_2-\lambda_1}{2}$ $\quad z = \dfrac{\lambda_1}{3}$

Cases:

	(1) B, (2) B	(1) B, (2) NB	(1) NB, (2) B	(1) NB, (2) NB
C:	$x-y+2z=3$ $x+y=3$	$x-y+2z=3$ $x+y>3$	$x-y+2z>3$ $x+y=3$	$x-y+2z>3$ $x+y>3$
CSC:	$\lambda_1\geq 0$ $\lambda_2\geq 0$	$\lambda_1\geq 0$ $\lambda_2=0$	$\lambda_1=0$ $\lambda_2\geq 0$	$\lambda_1=0$ $\lambda_2=0$
FOC:	$x=\frac{\lambda_1+\lambda_2}{4}$ $y=\frac{\lambda_2-\lambda_1}{2}$ $z=\frac{\lambda_1}{3}$	Same in all 4 cases		$x=0, y=0, z=0$ $x-y+2z=0 \not> 3$ not admissible $\Downarrow$ No candidates
	One candidate: $x=2, y=1, z=1$ $\lambda_1=3, \lambda_2=5$ $f=12$	$x=\frac{\lambda_1}{4}$ $y=-\frac{\lambda_1}{2}$ $z=\frac{\lambda_1}{3}$ $17\lambda_1=36$ $\lambda_1=\frac{36}{17}$ $x=\frac{9}{17}$ $y=-\frac{18}{17}$ $x+y<0$ not admissible $\Downarrow$ no candidates	$x=\frac{\lambda_2}{4}$ $y=\frac{\lambda_2}{2}$ $z=0$ $3\lambda_2=12$ $\lambda_2=4$ $x=1, y=2,$ $z=0, \lambda_1=0,$ $\lambda_2=4$ $1-2+2\cdot 0=-1 \not> 3$ not adm. no cad.	

$$\frac{\lambda_1+\lambda_2}{4}-\frac{\lambda_2-\lambda_1}{2}+2\cdot\frac{\lambda_1}{3}=3 \quad \cdot 12$$

$$3\lambda_1+3\lambda_2-6\lambda_2+6\lambda_1+8\lambda_1=36$$

$$\boxed{17\lambda_1-3\lambda_2=36}$$

$$16\lambda_1=48 \quad \underline{\lambda_1=3} \quad \underline{\lambda_2=5}$$

$$\frac{\lambda_1+\lambda_2}{4}+\frac{\lambda_2-\lambda_1}{2}=3 \quad \cdot 8$$

$$2\lambda_1+2\lambda_2+4\lambda_2-4\lambda_1=24$$

$$-2\lambda_1+6\lambda_2=24$$

$$\boxed{-\lambda_1+3\lambda_2=12}$$

Conclusion:

Only one cand. that satisfies FOC + C + CSC:

$$\boxed{(x,y,z;\lambda_1,\lambda_2)=(2,1,1;3,5) \qquad f=12}$$

NDCQ fails:

$g_1 = -(x-y+2z) = -x+y-2z$

$g_2 = -(x+y) = -x-y$

$$\begin{pmatrix} -1 & 1 & -2 \\ -1 & -1 & 0 \end{pmatrix}$$

NDCQ:

(1) B, (1) B: $\mathrm{rk}\begin{pmatrix} -1 & \boxed{\begin{matrix} 1 & -2 \\ -1 & 0 \end{matrix}} \\ -1 & \end{pmatrix} = 2$ ok

(1) B, (2) NB: $\mathrm{rk}\begin{pmatrix} -1 & 1 & \boxed{-2} \end{pmatrix} = 1$ ok

(1) NB, (2) B: $\mathrm{rk}\begin{pmatrix} -1 & -1 & 0 \end{pmatrix} = 1$ ok

(1) NB, (2) NB: no condition ok

No points (adm) where NDCQ fails.

two possibilities:

$(x,y,z) = (2,1,1)$ is minimum of f (max. of $-f$)

there is no minimum of f (max. of $-f$)

(2), (3) Lagrange / Kuhn - Tucker SOC

" (second order conditions)

Candidates:

Langrange: Lagrange conditions (FOC + C)
Kuhn-tucker: Kuhn-Tucker conditions (FOC + C + CSC)

+

L/KT: Admissible pts where NDCQ fails

Best candidate: Compute f for each candidate and compare.

How to determine if the best candidate is max/min:

i) **SOC:** Second order condition

ii) **Extreme value thm:** If D is bounded (D = set of admissible pts) then the best candidate is max/min

iii) Try 'something else.

SOC: (convexity / concavity)

Thm:

If $(x_1^*, x_2^*, \dots, x_n^*; \lambda_1^*, \lambda_2^*, \dots \lambda_m^*)$ satisfies FOC + C (in the Lagrange case) or FOC + C + CSC (in the Kuhn-Tucker case), then consider the function:

$$\mathcal{L} = \mathcal{L}(x_1, x_2, \dots, x_n; \lambda_1^*, \lambda_2^*, \dots, \lambda_m^*)$$

If $\mathcal{L}(\underline{x}; \underline{\lambda}^*)$ is <u>convex</u>, then $\underline{x}^*$ is a <u>min</u>.

If $\mathcal{L}(\underline{x}; \lambda^*)$ is <u>concave</u>, then $\underline{x}^*$ is a <u>max</u>.

<u>Ex.</u> $\max\ -(2x^2 + y^2 + 3z^2)$

when $x - y + 2z \geq 3$

$x + y \geq 3$

<u>Best candidate for max for $-f$:</u>

$x = 2,\ y = 1,\ z = 1,\ \lambda_1 = 3,\ \lambda_2 = 5$

$\mathcal{L}(x, y, z; 3, 5) = -2x^2 - y^2 - 3z^2 + 3(x - y + 2z) + 5(x + y)$

$$H(\mathcal{L}) = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

$D_1 = -4 < 0$

$D_2 = 8 > 0$

$D_3 = -48 < 0$

$H(\mathcal{L})$ neg. defn. everywhere $\Rightarrow \mathcal{L}$ is concave

<u>Conclusion:</u>

$(x, y, z) = (2, 1, 1)$ is <u>max</u> for $-f$

$\Leftarrow$

<u>min</u> for f

Problem: $\max\ 2x^2+y^2+3z^2$ when $\begin{array}{l} x-y+2z \geq 3 \\ x+y \geq 3 \end{array}$

$$= \max\ 2x^2+y^2+3z^2 \quad \text{when} \quad \begin{array}{l} -(x-y+2z) \leq -3 \\ -(x+y) \leq -3 \end{array}$$

$$\mathcal{L} = 2x^2+y^2+3z^2+\lambda_1(x-y+2z)+\lambda_2(x+y)$$

FOC:

$$\mathcal{L}'_x = 4x + \lambda_1+\lambda_2 = 0$$
$$\mathcal{L}'_y = 2y - \lambda_1+\lambda_2 = 0$$
$$\mathcal{L}'_z = 6z + 2\lambda_1 = 0$$
$$x = -\frac{\lambda_1+\lambda_2}{4} \ \ldots$$

previous case

$$-4x+\lambda_1+\lambda_2=0$$
$$-2y-\lambda_1+\lambda_2=0$$
$$-6z+2\lambda_1=0$$
$$x = \frac{\lambda_1+\lambda_2}{4}$$

$\Downarrow$

Concl: Solutions of FOC + C + CSC

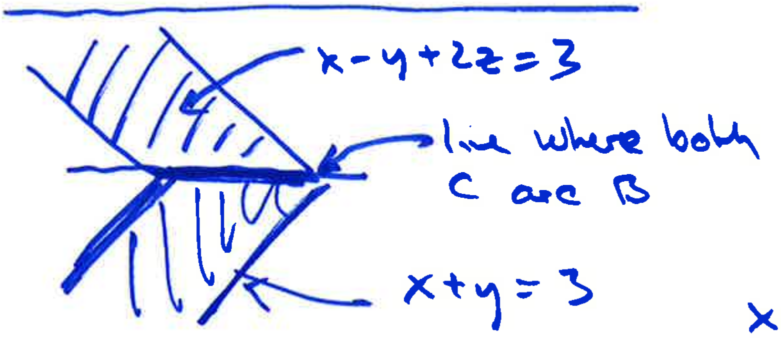
not admissible $\lambda_1<0,\ \lambda_2<0$ $\longrightarrow$ $x=2,\ y=1,\ z=1;\ \lambda_1=-3,\ \lambda_2=-5$

$\Downarrow$

No candidates

There is no max

Geometric interpretation:



$$\begin{array}{l} x-y+2z=3 \\ x+y=3 \end{array} \Big]^{-1}$$

$$x-y+2z=3$$
$$2y-2z=0$$

$x=3-z$, $y=z$, z free

$$f(x,y,z) = 2x^2 + y^2 + 3z^2$$

$$\left.\begin{aligned} x &= 3-z \\ y &= z \\ z &= z \quad (\text{free}) \end{aligned}\right\} \longrightarrow \begin{aligned} f &= 2\cdot(3-z)^2 + z^2 + 3z^2 \\ &= 2\cdot(9-6z+z^2) + z^2 + 3z^2 \\ &= 6z^2 - 12z + 18 \end{aligned}$$

$$z \to \pm\infty \implies f \to \infty$$

There is <u>no max</u>.

Alt: method shows that there is no max.

(we found a curve inside D = adm. pts where $f \to \infty$)