LECTURE 8 (B)

OCT. 9, 2014

GRA 6035 MATHEMATICS

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Plan:

1. Lagrange multipliers
2. Lagrange problem and SOC's
3. Kuhn-Tucker problems and SOC's

Reading:

[ME] 19.1, 19.4, (18.1 - 18.7)

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1. Lagrange multipliers

Ex: max $3x+4y$ when $x^2+y^2=25$

$\mathcal{L} = 3x+4y-\lambda\cdot(x^2+y^2)$

λ: Lagrange multiplier

FOC:
$\mathcal{L}'_x = 3-\lambda\cdot 2x = 0$

$\mathcal{L}'_y = 4-\lambda\cdot 2y = 0$

C: $x^2+y^2 = 25$

$x = \dfrac{3}{2\lambda}$ $(\lambda \neq 0)$

$y = \dfrac{4}{2\lambda}$

$$\left(\frac{3}{2\lambda}\right)^2 + \left(\frac{4}{2\lambda}\right)^2 = 25$$

$$\frac{9}{(2\lambda)^2} + \frac{16}{(2\lambda)^2} = \frac{25}{(2\lambda)^2} = 25$$

$(2\lambda)^2 = 1$ $\quad 2\lambda = \pm 1$

$\lambda = \pm 1/2$

Candidates for max:

$\lambda = 1/2$: $x=3,\ y=4,\ \lambda=1/2$

$\lambda = -1/2$: $x=-3,\ y=-4,\ \lambda=-1/2$

$f(3,4) = 25$ $\qquad f(-3,-4) = -25$

Candidates for max:

1) Solutions of FOC+C : $(x,y;\lambda)=(3,4;\frac{1}{2})$ $f=25$

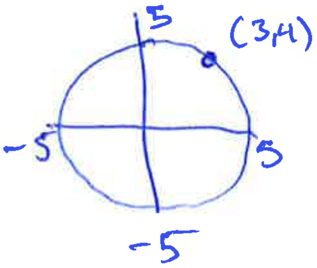
$(-3,-4;-\frac{1}{2})$ $f=-25$

2) NDCQ fails: No candidates.

NDCQ: $\operatorname{rk}\begin{pmatrix}2x & 2y\end{pmatrix} = 1$

$(g=x^2+y^2)$ NDCQ fails only if $x=y=0$, and this point is not admissible. NDCQ holds for all adm. pts. $(x^2+y^2=25)$

Since $x^2+y^2=25$ defines a bounded set, the extreme value thm. tells out that there is a max



Therefore $(x,y)=(3,4)$ is the max pt, and $f=25$ is max. value.

Interpretation of Lagrange multipliers

Gradient of f: $\nabla f = \begin{pmatrix} f'_{x_1} \\ f'_{x_2} \\ \vdots \\ f'_{x_n} \end{pmatrix}$

Gradient of g: $\nabla g = \begin{pmatrix} g'_{x_1} \\ \vdots \\ g'_{x_n} \end{pmatrix}$

FOC:

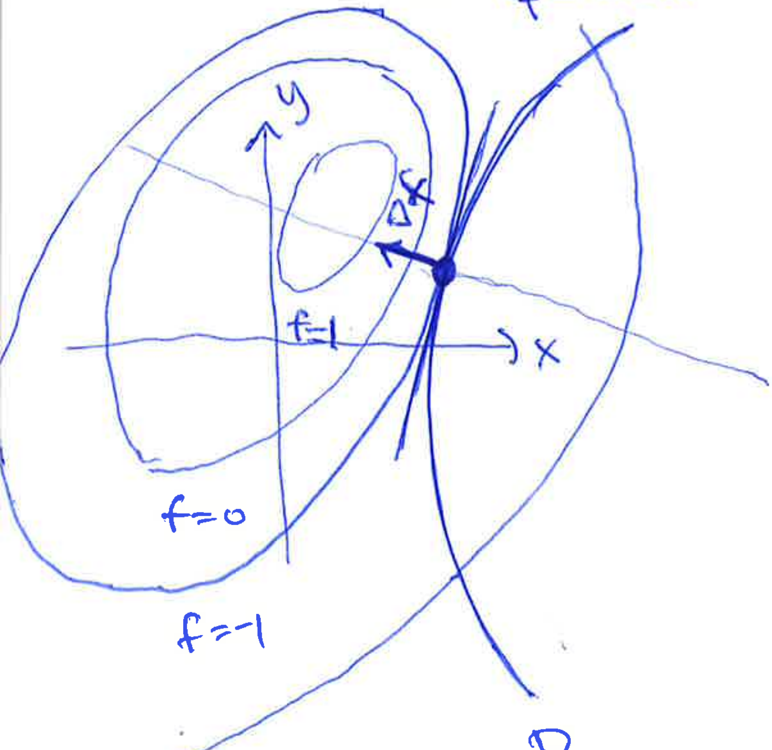
$\nabla f = \lambda \cdot \nabla g$

(gradients lie along the same line)

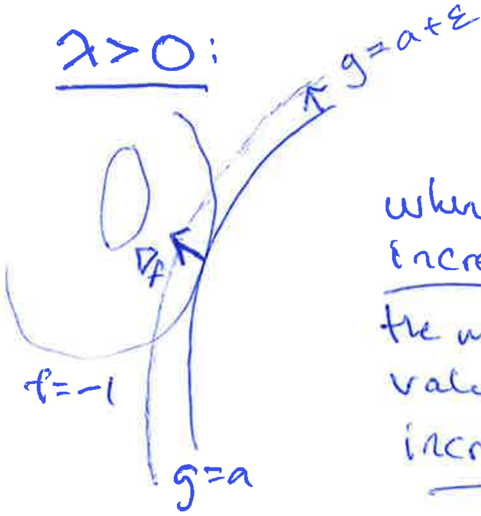
FOC:

$$\left.\begin{aligned} \mathcal{L}'_{x_1} &= f'_{x_1} - \lambda \cdot g'_{x_1} = 0 \\ \mathcal{L}'_{x_2} &= f'_{x_2} - \lambda \cdot g'_{x_2} = 0 \\ &\vdots \\ \mathcal{L}'_{x_n} &= f'_{x_n} - \lambda \cdot g'_{x_n} = 0 \end{aligned}\right\} \quad \nabla f - \lambda \cdot \nabla g = 0$$

∇f = the direction in which f increases at the quickest rate



$\lambda > 0$:



when we increase a, the maximum value will increase

$\lambda < 0$:

when we increase a, the maximum value will decrease

Result:

Let $\{\max f(\underline{x}) \text{ subj. to } g(\underline{x}) = a\}$ be a Lagrange problem with solution $x^*(a)$ that satisfy FOC+C. Then

$$\boxed{\frac{d}{da} f^*(a) = \lambda^*(a)}$$

where $\lambda^*(a)$ is the value of λ that together with $x^*(a)$ satisfy FOC+C, and $f^*(a) = f(x^*(a))$.
(the optimal value function)

In the example:

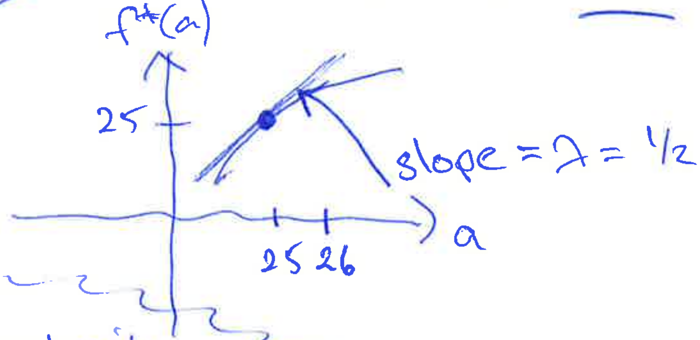
$a = 25$: $\quad x^*(25) = (3,4) \qquad f^*(25) = 25$

$h = (26-25) = 1$

$\lambda = 1/2$ tells us that $f^*(26) \simeq f(25) + 1 \cdot 1/2$

$= 25 + 1/2 = \underline{25.5}$

In general,
$f^*(a+h) \simeq f^*(a) + h \cdot \lambda^*(a)$
when h is small



If there are several constraints

$$\left.\begin{array}{l} g_1(\underline{x}) = a_1 \\ g_2(\underline{x}) = a_2 \\ \vdots \\ g_m(\underline{x}) = a_m \end{array}\right\} \qquad \frac{\partial f^*(a_1, \ldots, a_n)}{\partial a_i} = \lambda_i^*(a)$$

In Kuhn-Tucker problems, the interpretation of $\lambda_1, \ldots, \lambda_m$ is similar.

Ex: min $2x^2 + y^2 + 3z^2$ when $\begin{cases} x - y + 2z \geq 3 \\ x + y \geq 3 \end{cases}$

$\quad\;\; \overset{\shortparallel}{f(x,y,z)}$

KT problem in std form $\begin{cases} \max\ -(2x^2+y^2+3z^2) & \text{when } -(x-y+2z) \leq -3 \\ \quad\;\; \overset{\shortparallel}{-f(x,y,z)} & \quad\; -(x+y) \leq -3 \end{cases}$

$$\mathcal{L} = -(2x^2+y^2+3z^2) - \lambda_1\left(-(x-y+2z)\right) - \lambda_2\left(-(x+y)\right)$$
$$= -2x^2 - y^2 - 3z^2 + \lambda_1(x-y+2z) + \lambda_2(x+y)$$

FOC:
$$\mathcal{L}'_x = -4x + \lambda_1 + \lambda_2 = 0$$
$$\mathcal{L}'_y = -2y - \lambda_1 + \lambda_2 = 0$$
$$\mathcal{L}'_z = -6z + 2\lambda_1 = 0$$

C: $x - y + 2z \geq 3$
$x + y \geq 3$

CSC: $\lambda_1 \geq 0$ and $\lambda_1 \cdot (x - y + 2z - 3) = 0$
$\lambda_2 \geq 0$ " $\lambda_2 \cdot (x + y - 3) = 0$

Solve FOC + C + CSC:

FOC: $x = \dfrac{\lambda_1 + \lambda_2}{4}$ $\quad y = \dfrac{\lambda_2 - \lambda_1}{2}$ $\quad z = \dfrac{\lambda_1}{3}$

Cases:

(1) non-b. (2) non-b.	(1) non-bind. (2) bind.	(1) binding (2) non-bind.	(1) binding (2) binding
$x-y+2z>3$ $x+y>3$ $\lambda_1=0$ $\lambda_2=0$	$x-y+2z>3$ $x+y=3$ $\lambda_1=0$ $\lambda_2\geq 0$	$x-y+2z=3$ $x+y>3$ $\lambda_1\geq 0$ $\lambda_2=0$	$x-y+2z=3$ $x+y=3$ $\lambda_1\geq 0$ $\lambda_2\geq 0$
$x=0$ $y=0$ $z=0$ $\lambda_1=0$ $\lambda_2=0$ not admissible $\Downarrow$ no candidates	$x=\frac{\lambda_2}{4}$ $y=\frac{\lambda_2}{2}$ $z=0$ $\frac{\lambda_2}{4}+\frac{\lambda_2}{2}=3 \quad \vert\cdot 4$ $\lambda_2+2\lambda_2=12$ $\lambda_2=4$ $\lambda_1=0$ $x=1,\ y=2,\ z=0$ $x-y+2z=-1$ not adm. no cand.	no cand. (do the computations) $x=\frac{\lambda_1}{4}$ $y=-\frac{\lambda_1}{2}$ $z=\frac{\lambda_1}{3}$ $\frac{\lambda_1}{4}-\left(-\frac{\lambda_1}{2}\right)+\frac{2\lambda_1}{3}=3 \quad \vert\cdot 12$ $3\lambda_1+6\lambda_1+8\lambda_1=36$ $17\lambda_1=36 \quad \lambda_1=\frac{36}{17}$ $x+y=\frac{\lambda_1-2\lambda_1}{4}=-\frac{\lambda_1}{4}$ not adm.	$x=\frac{\lambda_1+\lambda_2}{4}$ $y=\frac{\lambda_2-\lambda_1}{2}$ $z=\frac{\lambda_1}{3}$ $\lambda_1=3,\ \lambda_2=5$ $x=2,\ y=1,\ z=1$ $f=8+1+3=12$

$$\frac{\lambda_1+\lambda_2}{4}-\frac{\lambda_2-\lambda_1}{2}+\frac{2\cdot\lambda_1}{3}=3 \quad \Big\vert\cdot 12$$

$$3\lambda_1+3\lambda_2-6\lambda_2+6\lambda_1+8\lambda_1=36$$

$$17\lambda_1-3\lambda_2=36$$

$$16\lambda_1=48$$

$$\lambda_1=3 \qquad \lambda_2=5$$

$\cdot 4$

$$\frac{\lambda_1+\lambda_2}{4}+\frac{\lambda_2-\lambda_1}{2}=3$$

$$\lambda_1+\lambda_2+2\lambda_2-2\lambda_1=12$$

$$-\lambda_1+3\lambda_2=12$$

Concl:

FOC + C + CSC give one candidate for min:
$$\underbrace{\text{FOC} + \text{C} + \text{CSC}}_{\text{KT conditions}}$$

$x = 2,\ y = 1,\ z = 1\ ;\ \lambda_1 = 3\ \ \lambda_2 = 5$

$f = 12$

NDCQ: $g_1 = x - y + 2z$
$g_2 = x + y$

$$\begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & 0 \end{pmatrix}$$

$g_1 > 3$, $g_2 > 3$	$g_1 > 3$, $g_2 = 3$	$g_1 = 3$, $g_2 > 3$	$g_1 = 3$, $g_2 = 3$
no condition	$\mathrm{rk}(1\ 1\ 0) = 1$	$\mathrm{rk}(1\ -1\ 2) = 1$	$\mathrm{rk}\begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & 0 \end{pmatrix} = 2$
ok	ok	ok	$\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2 \neq 0$ ok.

Cond from NDCQ: NDCQ satisfied at all admissible pts.

Candidates for min: $(x, y, z; \lambda_1, \lambda_2) = (2, 1, 1; 3, 5)$ $f = 12$

②,③ Lagrange / Kuhn-Tucker SOC

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(second order conditions)

Candidates:

Langrange: Lagrange conditions (FOC + C)
Kuhn-tucker: Kuhn-Tucker conditions (FOC + C + CSC)

+

L/KT: Admissible pts where NDCQ fails

Best candidate: Compute f for each candidate and compare.

How to determine if the best candidate is max/min:

i) SOC: Second order condition
ii) Extreme value thm: If D is bounded (D = set of admissible pts) then the best candidate is max/min

iii) Try something else.

SOC: Convexity / concavity

Thm:
If $(x_1^*, x_2^*, \dots, x_n^*; \lambda_1^*, \lambda_2^*, \dots, \lambda_m^*)$ satisfies either FOC+C (in Lagrange case) or FOC+C+CSC (in Kuhn-Tucker case), consider the function

$$\mathcal{L}(x_1, x_2, \dots, x_n; \lambda_1^*, \lambda_2^*, \dots, \lambda_m^*)$$

If $\mathcal{L}(x_1, \dots, x_n; \lambda_1^*, \dots, \lambda_m^*)$ is convex, then the pt. $(x_1^*, \dots, x_n^*)$ is a min. If $\mathcal{L}(x_1, \dots, x_n; \lambda_1^*, \dots, \lambda_m^*)$ is concave, then $(x_1^*, \dots, x_n^*)$ is a max.

Ex: min $2x^2+y^2+3z^2$ when $\begin{matrix} x-y+2z \geq 3 \\ x+y \geq 3 \end{matrix}$

$= \max -(2x^2+y^2+3y^2)$ when $\begin{cases} -x+y-2z \leq -3 \\ -x-y \leq -3 \end{cases}$

Best candidate: $(x,y,z;\lambda_1,\lambda_2) = (2,1,1;3,5)$

$$\mathcal{L}(x,y,z;3,5) = -2x^2-y^2-3z^2+3(x-y+2z) + 5(x+y)$$

$$H(\mathcal{L}) = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

$D_1 = -4 < 0$
$D_2 = 8 > 0$
$D_3 = -48 < 0$

negative defn. for all (x,y,z)
$\Downarrow$
$\mathcal{L}(x,y,z;3,5)$ is concave

$(x,y,z) = (2,1,1)$ is max for $-f$

i.e. — " — is min for f $\Longleftarrow$

Problem: Try to do the variation

$$\boxed{\max\ 2x^2+y^2+3z^2 \text{ when } \begin{matrix} x-y+2z \geq 3 \\ x+y \geq 3 \end{matrix}}$$

(with max instead of min).

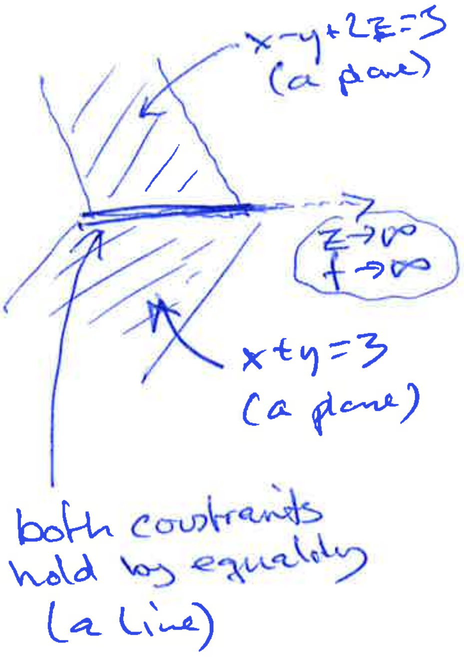
Answer: No candidates from FOC + C + CSC.
NDCQ satisfied for all adm. pts
$\Downarrow$
no candidates for max
$\Downarrow$
no max

We can also see this in the following way: When $\begin{cases} x-y+2z=3 \\ x+y = 3 \end{cases}$ we can solve the linear system, and get $\begin{cases} x = 3-z \\ y = z \\ z = z \text{ (free)} \end{cases}$

When we put this into f, we get

$$f(x,y,z) = 2x^2+y^2+3z^2 = 2\cdot(3-z)^2+z^2+3z^2$$
$$= 2\cdot(9-6z+z^2)+4z^2 = 6z^2-12z+18$$

when $z\to\infty$ and $x=3-z,\ y=z$, $\underset{6z^2-12z+18}{\underset{\shortparallel}{f}}\to\infty$



This also shows that there is **no max** since there is a trajectory in D along which $f\to\infty$.