LECTURE 4 (F)

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MATHEMATICS

Plan:

1. Eigenvalues and eigenvectors
2. Diagonalization

Reading:

[HE] 23.1 – 23.4, 23.6 – 23.7, 23.9

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1. Eigenvalues and eigenvectors

A: $n \times n$ - matrix

Definition:

A number λ is called an eigenvalue for A if the linear system

$$A \cdot \underline{v} = \lambda \cdot \underline{v}$$

has non-trivial solutions, $\underline{v} \neq \underline{0}$. In that case all solutions $\underline{v}$ are called eigenvectors for A with eigenvalue λ.

Ex: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$A\underline{v} = \lambda \underline{v}$ $\qquad \underline{v} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+y \\ x+2y \end{pmatrix} \quad \longleftarrow \quad A\underline{v}$$

$$\lambda \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix} \quad \longleftarrow \quad \lambda \cdot \underline{v}$$

$A\underline{v} = \lambda \underline{v}$: $\begin{pmatrix} 2x+y \\ x+2y \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix}$

$$2x + y = \lambda x$$
$$x + 2y = \lambda y$$

$$2x - \lambda x + y = 0$$
$$x + 2y - \lambda y = 0$$

$$(2-\lambda)x + y = 0$$
$$x + (2-\lambda)y = 0$$

Two possibilities:

$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} \neq 0$: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the unique solution

$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$: infinitely many solutions for $\begin{pmatrix} x \\ y \end{pmatrix}$

General method: $\quad A$: $n \times n$-matrix

$$A \cdot \underline{v} = \lambda \cdot \underline{v}$$

$$A \cdot \underline{v} - \lambda \underline{v} = \underline{0}$$

$$A\underline{v} - \lambda I \underline{v} = \underline{0}$$

$$(A - \lambda I)\underline{v} = 0$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \rightsquigarrow (A-\lambda \underline{I})\underline{v} = 0 \text{ has the form}$$

$$\begin{pmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{pmatrix} \underline{v} = \underline{0}$$

Characteristic equation: $\quad |A - \lambda I| = 0$

Eigenvalues of A = solutions of the char. eqn.

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33}-\lambda & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

$$(-\lambda)^n + \text{lower degree terms} = 0$$

polynomial eqn of order n

Summary:

Characteristic equation $|A-\lambda I|=0$

a) Eigenvalues of A = Solutions of the characteristic eqn. for λ.

b) Eigenvectors of A:

For each eigenvalue λ from a), solve the linear system $(A-\lambda I)\cdot \underline{v}=\underline{0}$ for $\underline{v}$.

(Gaussian elimination if $n>2$)

In general, it is difficult to solve the characteristic equation.

Ex: the case $n=2$ $\det(A-\lambda I)=0$

$$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix}=0 \quad \begin{pmatrix}\text{Char.}\\ \text{eqn.}\end{pmatrix}$$

$$(a-\lambda)(d-\lambda)-bc=0$$

$$\lambda^2-(a+d)\lambda+(ad-bc)=0$$

$$\boxed{\lambda^2-tr(A)\cdot\lambda+\det(A)=0}$$

$tr(A)=a+d$
$\det(A)=ad-bc$

tr = trace
(sum of elements on the diagonal)

Ex: $A=\begin{pmatrix} 7 & 1 \\ 1 & 7 \end{pmatrix}$ $\lambda^2-14\lambda+48=0$
(char. eqn.)

Fact:

$$|A-\lambda I| = (\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_r)\cdot Q(\lambda)$$

where $r \leq n$ and $Q(\lambda)=0$ has no sol's.

Possibilities:

i) $r=n$ and $Q(\lambda)$ is a non-zero constant

$\Rightarrow \lambda_1, \lambda_2, \ldots, \lambda_n$ <u>n eigenvalues</u>

ii) $r<n$ and $Q(\lambda)$ is <u>not</u> a constant

$\Rightarrow \lambda_1, \lambda_2, \ldots, \lambda_r$ and $Q(\lambda)=0$ (no sol's)

$\uparrow$

(typically $Q(\lambda) = \lambda^2+1$)

Ex: $A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix} \rightarrow \begin{vmatrix} -\lambda & -1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$

$$+(3-\lambda)\cdot\begin{vmatrix} -\lambda & -1 \\ +1 & -\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\cdot(\lambda^2+1) = 0$$

$$-(\lambda-3)\cdot(\lambda^2+1) = 0$$

<u>$\lambda=3$</u> or $\lambda^2+1=0$

<u>Only one eigenvalue</u>

Ex: $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$\lambda^2 - 0 \cdot \lambda + 1 = 0$$
$$\lambda^2 + 1 = 0$$

no solutions $\Rightarrow$ no eigenvalues (among real numbers)

$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$$\lambda^2 - 2\lambda + 1 = 0$$
$$(\lambda - 1)^2 = 0$$
$$(\lambda - 1) \cdot (\lambda - 1) = 0$$

$\lambda = 1$ (multiplicity two)

$$\lambda = \frac{2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 1}}{2} = \frac{2 \pm 0}{2}$$

$\lambda_1 = 1 \quad \lambda_2 = 1$

Ex: $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$

$$\begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 3-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$+(3-\lambda) \cdot \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$(3-\lambda) \cdot (\lambda^2 - 4\lambda + 3) = 0$$

$\lambda_1 = 3$ or $\lambda^2 - 4\lambda + 3 = 0$

$\lambda_2 = 1$, $\lambda_3 = 3$

$\lambda = 3$ multiplicity 2
$\lambda = 1$ mult. 1

$$-(\lambda - 3) \cdot (\lambda - 1)(\lambda - 3) = 0$$

Ex: $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ $\lambda_1 = \lambda_2 = 3$, $\lambda_3 = 1$

Eigenvectors:

$\lambda = 3$:

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-x + z = 0$

$\cancel{0 = 0}$

$\cancel{x - z = 0}$

$x = z$
y free
z free

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} z \\ 0 \\ z \end{pmatrix}$$

$$= y \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$\underline{v_1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\underline{v_2} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = 1$: $A - \lambda I$ for $\lambda = 1$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x + z = 0$
$2y = 0$
$\cancel{x + z = 0}$

$x = -z$
$y = 0$
z is free

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ 0 \\ z \end{pmatrix} = z \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

all multiples of

$\underline{v_3} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

In general, if there are k free variables $s_1, s_2, \dots, s_k$ the solutions can be written

$$s_1 \cdot \underline{v_1} + s_2 \cdot \underline{v_2} + \dots + s_k \cdot \underline{v_k}$$

The vectors $\{\underline{v_1}, \dots, \underline{v_k}\}$ are linearly independent.

Fact:

If λ is an eigenvalue for A of multiplicity m, then the linear system $(A-\lambda I)\underline{v} = \underline{0}$ has at least one and at most m degrees of freedom.

Ex: $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ $\lambda_1 = \lambda_2 = 3$, $\lambda_3 = 1$

$\lambda = 3$: Eigenvectors $y \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$(m=2)$

2 degrees of freedom

Ex: $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\lambda_1 = \lambda_2 = 1$

$\lambda = 1$: Eigenvectors: $\begin{pmatrix} \overset{=0}{1-1} & 1 \\ 0 & \underset{=0}{1-1} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$(m=2)$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} = \underline{x \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$

$y = 0$

$\cancel{0 = 0}$

x free

$y = 0$

1 degree of freedom

Fact:

If A has n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then

$$\det(A) = \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n$$

$$\operatorname{tr}(A) = \lambda_1 + \lambda_2 + \ldots + \lambda_n$$

2 Diagonalization

Definition:

An $n \times n$-matrix A is diagonalizable if there is an invertible matrix P such that

$$P^{-1}AP = D$$

where D is a diagonal matrix.

Fact:

A diagonalizable $\Longleftrightarrow$ A has n linearly independent eigenvectors

Why? If $\{\underline{v_1}, \dots, \underline{v_n}\}$ eigenvectors, let us look at:

$$P = \left(\underline{v_1} \,\middle|\, \underline{v_2} \,\middle|\, \cdots \,\middle|\, \underline{v_n} \right)$$

P invertible
$\Updownarrow$
$\{\underline{v_1}, \dots, \underline{v_n}\}$ lin. independent

$$\begin{aligned} A \cdot P &= A \cdot \left(\underline{v_1} \,|\, \underline{v_2} \,|\, \dots \,|\, \underline{v_n} \right) \\ &= \left(A\underline{v_1} \,|\, A\underline{v_2} \,|\, \dots \,|\, A\underline{v_n} \right) \\ &= \left(\lambda_1 \underline{v_1} \,|\, \lambda_2 \underline{v_2} \,|\, \dots \,|\, \lambda_n \underline{v_n} \right) = P \cdot \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \end{pmatrix} \end{aligned}$$

$$\boxed{AP = PD}$$

$$\Downarrow$$

$$P^{-1}AP = D$$

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

Ex: $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$

$\underbrace{\lambda_1 = \lambda_2 = 3 \ , \ \lambda_3 = 1}_{n=3}$ eigenvalues (counted with multiplicity)

$P = \Big(\underbrace{\underline{v_1} \,\Big|\, \underline{v_2}}_{\lambda = 3} \,\Big|\, \underbrace{\underline{v_3}}_{\lambda=1} \Big)$ $\qquad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\lambda = 3$: $\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad \underline{v_1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$= y \cdot \underline{v_1} + z \cdot \underline{v_2}$$

$\lambda = 1$: $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ 0 \\ z \end{pmatrix} = z \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{}$ $\qquad \underline{v_3} \shortparallel$

$P = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ ← The column vectors are linearly independent:

* eigenvectors corresp. to different eigenvalues are lin. indep.

Concl: A is diagonalizable.

$$\underbrace{\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}^{-1}}_{P^{-1}} \cdot \underbrace{\begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}}_{P} = \underbrace{\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{D}$$

When is A diagonalizable?

A diagonalizable $\Longleftrightarrow$ $\begin{cases} \text{i) There are } n \text{ eigenvalues } \lambda_1, \lambda_2, \dots, \lambda_n \text{ of } A \\ \quad \textbf{and} \\ \text{ii) For each eigenvalue } \lambda \text{ with multiplicity } m, \\ (A - \lambda I)\underline{v} = \underline{0} \text{ has } m \text{ degrees of freedom.} \end{cases}$

Ex: $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ <u>not</u> diagonalizable (no eigenvalues)

$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ <u>not</u> — " —

$\lambda_1 = \lambda_2 = 1$

$D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$P = \begin{pmatrix} 1 & ? \\ 0 & ? \end{pmatrix}$

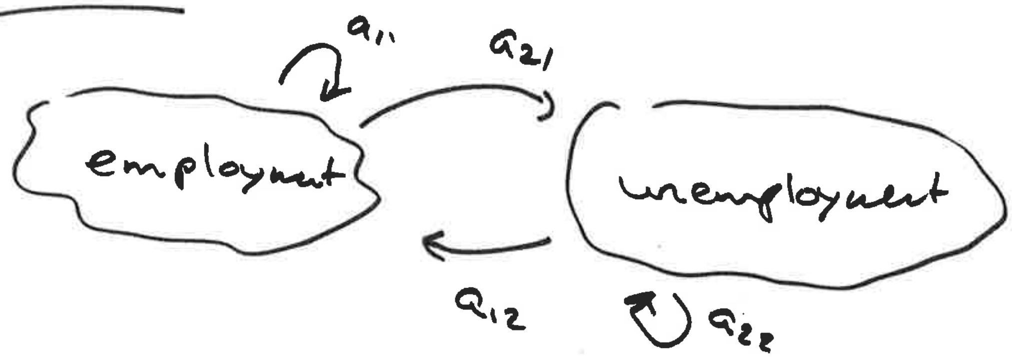
$\uparrow$

$(A - \lambda I)\underline{v} = \underline{0}$ for $\lambda = 1$ you get <u>one</u> degree of freedom

Fact:

1) If A ~~is~~ symmetric, then A is diagonalizable
2) If A has n distinct eigenvalues, then A is diagonalizable.

Application: Markov chains



State vector: $\begin{pmatrix} e \\ u \end{pmatrix}$ $\quad$ e = share of employment
u = — " — unemployment

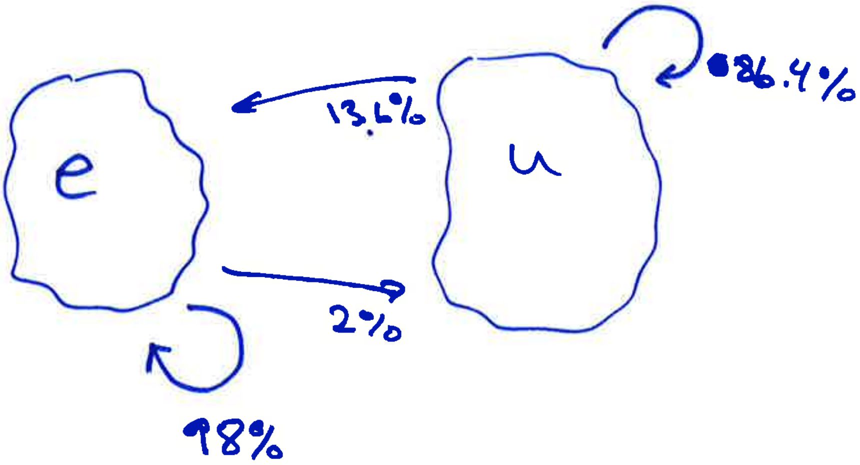
transition matrix: $\begin{pmatrix} e_{t+1} \\ u_{t+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} e_t \\ u_t \end{pmatrix}$

$$e_{t+1} = a_{11} e_t + a_{12} u_t$$
$$u_{t+1} = a_{21} e_t + a_{22} u_t$$

Ex: $A = \begin{pmatrix} 0.98 & 0.136 \\ 0.02 & 0.864 \end{pmatrix}$

$\underline{v_0} = \begin{pmatrix} 0.9 \\ 0.1 \end{pmatrix}$

initial state
= 10% unemploym.



Long term:

$\underline{v_1} = A \cdot \underline{v_0}$ (after one week)

$\underline{v_2} = A \cdot \underline{v_1} = A \cdot (A \underline{v_0}) = A^2 \cdot \underline{v_0}$ (after two weeks)

$\vdots$

$\underline{v_n} = A^n \cdot \underline{v_0}$ (after n weeks)

What happens when $n \to \infty$ (long term)?

Eigenvalues and eigenvectors:

$$\begin{vmatrix} 0.98-\lambda & 0.136 \\ 0.02 & 0.864-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 1.844\lambda + 0.844 = 0$$

$$\lambda = \frac{1.844 \pm \sqrt{1.844^2 - 4 \cdot 0.844}}{2}$$

$\underline{\lambda_1 = 1}$ $\lambda_2 = \underline{0.844}$

$\underline{\lambda = 1}$: $\begin{pmatrix} -0.02 & 0.136 \\ 0.02 & -0.136 \end{pmatrix}$

$-0.02x + 0.136y = 0$

$x = \frac{0.136}{0.02} y = 6.8y$ (y free)

$\underline{v} = \begin{pmatrix} 6.8y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} 6.8 \\ 1 \end{pmatrix}$

$\Rightarrow v_1 = \begin{pmatrix} 6.8 \\ 1 \end{pmatrix}$

$\underline{\lambda = 0.844}$:

$\begin{pmatrix} 0.136 & 0.136 \\ 0.02 & 0.02 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$y = -x$, x free

$\underline{v} = \begin{pmatrix} x \\ -x \end{pmatrix} = x \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\underline{v_2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Conclusión:

A has eigenvalues $\lambda_1 = 1$, $\lambda_2 = 0.844$

and eigenvectors $\underline{v_1} = \begin{pmatrix} 6.8 \\ 1 \end{pmatrix}$ $\underline{v_2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 0.844 \end{pmatrix} \qquad P = \begin{pmatrix} 6.8 & 1 \\ 1 & -1 \end{pmatrix} \qquad P^{-1} = \frac{1}{-7.8}\begin{pmatrix} -1 & -1 \\ -1 & 6.8 \end{pmatrix}$$

$$= \frac{1}{7.8}\begin{pmatrix} 1 & 1 \\ 1 & -6.8 \end{pmatrix}$$

Can compute A^n:

$$P^{-1}AP = D \Rightarrow A = PDP^{-1} \Rightarrow A^n = (PD\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}})\cdots(\cancel{P}DP^{-1})$$

$$= P \cdot D^n \cdot P^{-1}$$

$$A^n = \begin{pmatrix} 6.8 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1^n & 0 \\ 0 & 0.844^n \end{pmatrix} \cdot \frac{1}{7.8}\begin{pmatrix} 1 & 1 \\ 1 & -6.8 \end{pmatrix}$$

When $n \to \infty$, $D^n \to \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and

$$A^n \to \begin{pmatrix} 6.8 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \frac{1}{7.8}\begin{pmatrix} 1 & 1 \\ 1 & -6.8 \end{pmatrix}$$

$$= \begin{pmatrix} 6.8 & 0 \\ 1 & 0 \end{pmatrix} \cdot \frac{1}{7.8} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -6.8 \end{pmatrix}$$

$$= \frac{1}{7.8} \cdot \begin{pmatrix} 6.8 & 6.8 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 6.8/7.8 & 6.8/7.8 \\ 1/7.8 & 1/7.8 \end{pmatrix}$$

$$\simeq \underline{\begin{pmatrix} 0.872 & 0.872 \\ 0.128 & 0.128 \end{pmatrix}}$$

Conclusion: In the long run $(n \to \infty)$, then state vector is

$$A^n \cdot \underline{v_0} \to \begin{pmatrix} 0.872 & 0.872 \\ 0.128 & 0.128 \end{pmatrix} \cdot \begin{pmatrix} 0.9 \\ 0.1 \end{pmatrix} = \underline{\begin{pmatrix} 0.872 \\ 0.128 \end{pmatrix}}$$

That is, unemployment is 12.8%