LECTURE 4 (13)

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GRA 6035 MATHEMATICS

Plan:

1. Eigenvalues and eigenvectors
2. Diagonalization

Reading:

[ME] 23.1-23.4, 23.6-23.7, 23.9

1 Eigenvalues and eigenvectors

A: $n \times n$-matrix

Definition

A number λ is called an eigenvalue for A if the linear system

$$A \cdot \underline{v} = \lambda \cdot \underline{v}$$

has non-trivial solutions $\underline{v} \neq \underline{0}$. In that case, all solutions $\underline{v}$ of this linear system are called eigenvectors for A with eigenvalue λ.

Ex:

$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$: $A \cdot \underline{v} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ x + 2y \end{pmatrix}$

$$\lambda \cdot \underline{v} = \lambda \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix}$$

$$\begin{pmatrix} 2x+y \\ x+2y \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix}$$

$$2x+y = \lambda x$$
$$x+2y = \lambda y$$

$$\boxed{\begin{aligned}(2-\lambda)x + y &= 0\\ x + (2-\lambda)y &= 0\end{aligned}}$$

$$2x - \lambda x + y = 0$$
$$x + 2y - \lambda y = 0$$

Nontrivial solutions $(x,y) \neq (0,0)$ $\iff \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$$(2-\lambda)\cdot(2-\lambda) - 1^2 = 0$$
$$\lambda^2 - 4\lambda + (4-1) = 0$$
$$\lambda = \frac{4 \pm \sqrt{4^2 - 4\cdot 1\cdot 3}}{2}$$
$$= \frac{4 \pm 2}{2}$$
$$\underline{\lambda = 3}, \quad \underline{\lambda = 1}$$

Conclusion:

The eigenvalues of $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ are $\lambda_1 = 3$, $\lambda_2 = 1$

Eigen vectors:

$\lambda_1 = 3$: $-x + y = 0$

$\cancel{x - y = 0}$

$y = x$, x free

$$\underline{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x \end{pmatrix} = x \cdot \underline{\underline{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}}$$

$\lambda_2 = 1$: $x + y = 0$

$\cancel{x + y = 0}$

$y = -x$, x free

$$\underline{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -x \end{pmatrix} = x \cdot \underline{\underline{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}}$$

Conclusion:

Eigenvectors with $\lambda = 3$: Multiples of $\underline{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Eigenvectors with $\lambda = 1$: Multiples of $\underline{v_2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

If you choose y free

$x = -y$, y free

$$\underline{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y \cdot \underline{\underline{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}}$$

General method: A $n\times n$-matrix

$A\cdot \underline{v} = \lambda\cdot \underline{v}$

$A\cdot \underline{v} - \lambda\cdot \underline{v} = \underline{0}$

$A\cdot \underline{v} - \lambda I\cdot \underline{v} = \underline{0}$

$$\boxed{(A-\lambda I)\cdot \underline{v} = \underline{0}}$$

$$A-\lambda I = \begin{pmatrix} a_{11} & a_{12} & \dots \\ a_{21} & a_{22} & \dots \\ \vdots & \vdots & \end{pmatrix} - \begin{pmatrix} \lambda & 0 & \dots \\ 0 & \lambda & \dots \\ \vdots & \vdots & \end{pmatrix} = \begin{pmatrix} a_{11}-\lambda & a_{12} & \dots \\ a_{21} & a_{22}-\lambda & \dots \\ \vdots & \vdots & \end{pmatrix}$$

Characteristic equation:

$$|A-\lambda I| = 0$$

(a) Eigenvalues of A = Solutions of the characteristic equation

(b) Eigenvectors of A with given eigenvalue λ = solutions of the linear system $(A-\lambda I)\cdot \underline{v} = \underline{0}$

Comment:

Part a): Solve polynomial equation of degree n $\to$ difficult

Part b): Solve linear system $\to$ easy (Gaussian elimination)

If there are k free variables $s_1, \dots, s_k$, then the solutions (= eigenvectors) can be written

$$\underline{v} = s_1 \underline{v_1} + s_2 \underline{v_2} + \dots + s_k \underline{v_k}$$

and $\{\underline{v_1}, \underline{v_2}, \dots, \underline{v_k}\}$ are linearly independent.

Ex I $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$\begin{vmatrix} 0-\lambda & 1 \\ -1 & 0-\lambda \end{vmatrix} = 0$$

$\lambda^2 + 1 = 0$

$\lambda^2 = -1$ no solutions (among real numbers)

$\Downarrow$

No (real) eigenvalues

Ex2: $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$$\begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = 0$$

$(1-\lambda)^2 - 0 = 0$

$(1-\lambda)\cdot(1-\lambda) = 0$ → $(1-\lambda)^2 = 0$

only one (double) root

$\lambda_1 = 1$ $\lambda_2 = 1$

(this is called multiplicity two)

Ex: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$|A - \lambda I| = 0 \rightarrow \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$

trace → $tr(A) = a+d$

$det(A) = ad - bc$

$(a-\lambda)(d-\lambda) - bc = 0$

$ad - a\lambda - d\lambda + \lambda^2 - bc = 0$

$\lambda^2 - (a+d)\lambda + (ad-bc) = 0$

$\lambda^2 - tr(A)\cdot\lambda + det(A) = 0$

Ex3: $A = \begin{pmatrix} 3 & 1 \\ -1 & 7 \end{pmatrix}$

$\lambda^2 - 10\lambda + 22 = 0$

$$\lambda = \frac{10 \pm \sqrt{10^2 - 4\cdot 22}}{2}$$

$$= \frac{10 \pm \sqrt{12}}{2} = 5 \pm \sqrt{3}$$

$\lambda_1 = 5 + \sqrt{3}$

$\lambda_2 = 5 - \sqrt{3}$

Characteristic equation:

$$|A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

$$= (-\lambda)^n + \dots \text{(terms of lower deg.)} = 0$$

Fact: $|A-\lambda I| = (\lambda-\lambda_1)(\lambda-\lambda_2)\dots(\lambda-\lambda_r)\cdot Q(\lambda) = 0$

where $Q(\lambda)=0$ has no solutions

Possibilities:

i) $Q(\lambda)$ is a non-zero constant $(r=n)$

$\Rightarrow$ there are n solutions

ii) $Q(\lambda)$ is not a constant $(r<n)$

$\Rightarrow$ there are $r<n$ solutions

Fact: 1) When A is symmetric then $r=n$.

2) $\left.\begin{array}{l} \det(A) = \lambda_1 \cdot \ldots \cdot \lambda_n \\ tr(A) = \lambda_1 + \dots + \lambda_n \end{array}\right\}$ when there are n solutions

Ex of possibility i) and ii):

i) $|A-\lambda I| = -(\lambda-2)(\lambda-4)(\lambda-10)$ $\Rightarrow$ Eigenvalues 2, 4, 10

ii) $|A-\lambda I| = -(\lambda-1)\cdot(\lambda^2+4)$ " 1 (no solution of $\lambda^2+4=0$)

Ex: $A = \begin{pmatrix} 7 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 7 \end{pmatrix}$ $n=3$

$$\begin{vmatrix} 7-\lambda & 0 & 3 \\ 0 & 2-\lambda & 0 \\ 3 & 0 & 7-\lambda \end{vmatrix} = 0$$

$$+(2-\lambda)\cdot \begin{vmatrix} 7-\lambda & 3 \\ 3 & 7-\lambda \end{vmatrix} = 0$$

If you multiply out, you get $-\lambda^3+16\lambda^2-68\lambda+80=0$. This is difficult to solve! Try to keep $\det(A-\lambda I)$ factorized if possible!

$$(2-\lambda)\cdot(\lambda^2-14\lambda+40)=0$$

$\lambda=2$ or $\lambda^2-14\lambda+40=0$

$$\lambda = \frac{14 \pm \sqrt{14^2-4\cdot 40}}{2} = \frac{14 \pm 6}{2} = 7 \pm 3$$

$\underline{\underline{\lambda_1 = 2}}$ $\underline{\underline{\lambda_2 = 10}}$ $\underline{\underline{\lambda_3 = 4}}$

<u>Factorization:</u> $-(\lambda-2)\cdot(\lambda-10)(\lambda-4)=0$

<u>Eigenvectors:</u>

If λ is an eigenvalue of A of multiplicity m, then the linear system

$$(A-\lambda I)\cdot \underline{v} = \underline{0}$$

has <u>at most</u> m degrees of freedom and <u>at least</u> one degree of freedom.

Ex: $A = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}$

$$\lambda^2 - 6\lambda + 9 = 0$$
$$(\lambda - 3)^2 = 0$$
$$\underline{\lambda_1 = 3}, \quad \underline{\lambda_2 = 3}$$

Eigenvectors:

$$\begin{pmatrix} 4-3 & -1 \\ 1 & 2-3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$x - y = 0$ $x = y$, y free

$\cancel{x - y = 0}$

degrees of freedom is <u>at least 1</u> and <u>at most 2</u> (since $\lambda = 3$ has mult. 2).

Computations show that # degrees of freedom $= 1$ in this case

$$\underline{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$= y \cdot \underline{v_1}, \quad \underline{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Eigenvectors $(\lambda = 3)$ = all multiples of $\underline{v_1}$.

(2) Diagonalization

Defn: An $n \times n$-matrix A is diagonalizable if there is a diagonal matrix D and an invertible matrix P such that

$$\boxed{P^{-1} A P = D}$$

Fact: If A has n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (counted with multiplicity) and if there are n linearly independent eigenvectors for A, $\underline{v_1}, \underline{v_2}, \ldots, \underline{v_n}$, then A is diagonalizable and we may take

$$P = \left(\underline{v_1} \,\middle|\, \underline{v_2} \,\middle|\, \cdots \,\middle|\, \underline{v_n} \right) \qquad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

Order of eigenvalues correspond to order of eigenvectors
$A \cdot \underline{v_i} = \lambda_i \cdot \underline{v_i}$

Ex: $A = \begin{pmatrix} 7 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 7 \end{pmatrix}$

Eigenvalues: $\lambda_1 = 2 \quad \lambda_2 = 4 \quad \lambda_3 = 10$ (we found these before)

Eigenvectors:

$\lambda = 2$: $\begin{pmatrix} 5 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 5 \end{pmatrix} \underline{x} = \underline{0}$

$\downarrow$

$\begin{pmatrix} 5 & 0 & 3 \\ 0 & 0 & 16/5 \\ 0 & 0 & 0 \end{pmatrix}$

one free var: x_2

$x_1 = 0$
x_2 free
$x_3 = 0$

$$\boxed{\underline{v} = \begin{pmatrix} 0 \\ x_2 \\ 0 \end{pmatrix} = x_2 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$$

Concl: A is diagonalizable (find $\underline{v_2}$, $\underline{v_3}$ in similar way)

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 10 \end{pmatrix}$$

$$P = \left(\underline{v_1} \,\middle|\, \underline{v_2} \,\middle|\, \underline{v_3} \right)$$

$\underline{v_1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \underline{v_2} \quad \underline{v_3}$

How to check if A is diagonalizable:

A: $n \times n$ - matrix

i) Compute eigenvalues of A: $\lambda_1, \lambda_2, .., \lambda_r$

Use multiplicity (ie. if $\lambda = 1$ has multiplicity 2, then $\lambda_1 = 1, \lambda_2 = 1$ is included twice)

If $r = n$ (there are n eigenvalues), then we take

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

and check ii), if $r < n$ then A is not diagonalizable.

ii) Compute eigenvectors for A:

If λ_i has multiplicity m_i, then there are two possibilities:

i) $(A - \lambda_i I)\underline{v} = \underline{0}$ has m_i degrees of freedom

Solutions: $\underline{v} = s_1 \cdot \underline{v_1} + s_2 \underline{v_2} + .. + s_{m_i} \cdot \underline{v_{m_i}}$
where $s_1 .., s_{m_i}$ are free var's, $\{\underline{v_1}, \underline{v_2}, .., \underline{v_{m_i}}\}$ eigenvectors (automatically lin. independent)

$\Rightarrow$ A diagonalizable (if this is the case for all eigenvalues)

$$P = \left(\underline{v_1} \mid \underline{v_2} \mid \dots \mid \underline{v_n} \right)$$

ii) $(A - \lambda_i I)\underline{v} = 0$ has less than m_i degrees of freedom:

A not diag.

(not enough eigenvectors)

Two eigenvectors for A corresponding to different eigenvalues are always linearly independent.

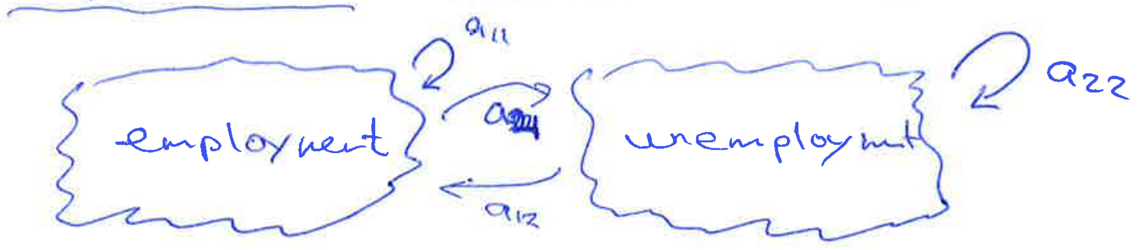
Facts:

1) If A is symmetric, then it is diagonalizable.

2) If A has n different (i.e all multiplicity 1) eigenvalues, then it is diagonalizable.

$$A \text{ diagonalizable} \iff \begin{cases} A \text{ has } n \text{ eigenvalues} \\ + \\ A \text{ has } n \text{ linearly independent eigenvectors} \end{cases}$$

Application: Markov chains



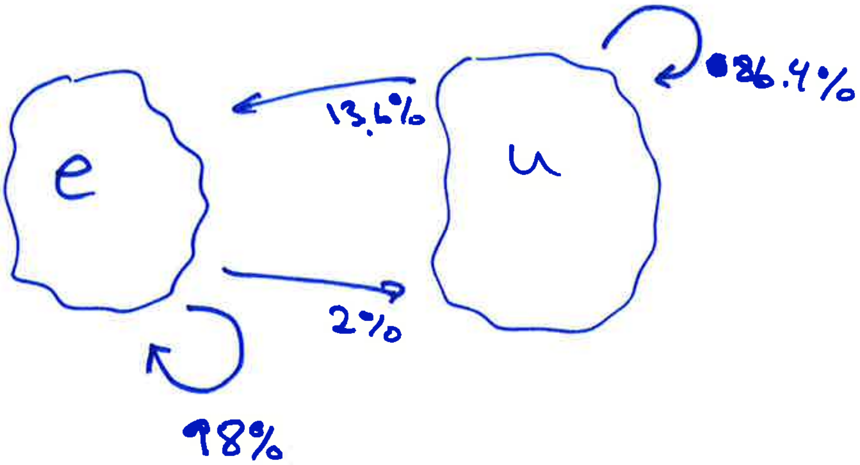
state vector: $\begin{pmatrix} e \\ u \end{pmatrix}$ $e =$ share of employed
$u =$ — " — unemployed

transition matrix: $\begin{pmatrix} e_{t+1} \\ u_{t+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} e_t \\ u_t \end{pmatrix}$

$$e_{t+1} = a_{11} e_t + a_{12} \cdot u_t$$
$$u_{t+1} = a_{21} e_t + a_{22} u_t$$

Ex: $A = \begin{pmatrix} 0.98 & 0.136 \\ 0.02 & 0.864 \end{pmatrix}$ $\quad \underline{v}_0 = \begin{pmatrix} 0.9 \\ 0.1 \end{pmatrix}$

initial state
$= 10\%$ unemploym.



Long term:

$\underline{v}_1 = A \cdot \underline{v}_0$ (after one week)

$\underline{v}_2 = A \cdot \underline{v}_1 = A \cdot (A \underline{v}_0) = A^2 \cdot \underline{v}_0$ (after two weeks)

$\vdots$

$\underline{v}_n = A^n \cdot \underline{v}_0$ (after n weeks)

What happens when $n \to \infty$ (long term)?

Eigenvalues and eigenvectors:

$$\begin{vmatrix} 0.98-\lambda & 0.136 \\ 0.02 & 0.864-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 1.844\lambda + 0.844 = 0$$

$$\lambda = \frac{1.844 \pm \sqrt{1.844^2 - 4 \cdot 0.844}}{2}$$

$\underline{\lambda_1 = 1}$ $\quad \lambda_2 = \underline{0.844}$

$\underline{\lambda = 1}$: $\begin{pmatrix} -0.02 & 0.136 \\ 0.02 & -0.136 \end{pmatrix}$

$-0.02x + 0.136y = 0$

$x = \frac{0.136}{0.02} y = 6.8y$ (y free)

$\underline{v} = \begin{pmatrix} 6.8y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} 6.8 \\ 1 \end{pmatrix}$

$\Rightarrow v_1 = \begin{pmatrix} 6.8 \\ 1 \end{pmatrix}$

$\underline{\lambda = 0.844}$:

$\begin{pmatrix} 0.136 & 0.136 \\ 0.02 & 0.02 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$y = -x$, x free

$\underline{v} = \begin{pmatrix} x \\ -x \end{pmatrix} = x \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\quad \underline{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Conclusión:

A has eigenvalues $\lambda_1 = 1$, $\lambda_2 = 0.844$
and eigenvectors $\underline{v_1} = \begin{pmatrix} 6.8 \\ 1 \end{pmatrix}$ $\underline{v_2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 0.844 \end{pmatrix} \quad P = \begin{pmatrix} 6.8 & 1 \\ 1 & -1 \end{pmatrix} \quad P^{-1} = \frac{1}{-7.8}\begin{pmatrix} -1 & -1 \\ -1 & 6.8 \end{pmatrix}$$
$$= \frac{1}{7.8}\begin{pmatrix} 1 & 1 \\ 1 & -6.8 \end{pmatrix}$$

Can compute A^n:

$$P^{-1}AP = D \Rightarrow A = PDP^{-1} \Rightarrow A^n = (PD\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}})\cdots(\cancel{P}DP^{-1})$$
$$= P \cdot D^n \cdot P^{-1}$$

$$A^n = \begin{pmatrix} 6.8 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1^n & 0 \\ 0 & 0.844^n \end{pmatrix} \cdot \frac{1}{7.8}\begin{pmatrix} 1 & 1 \\ 1 & -6.8 \end{pmatrix}$$

When $n \to \infty$, $D^n \to \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and

$$A^n \to \begin{pmatrix} 6.8 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \frac{1}{7.8}\begin{pmatrix} 1 & 1 \\ 1 & -6.8 \end{pmatrix}$$
$$= \begin{pmatrix} 6.8 & 0 \\ 1 & 0 \end{pmatrix} \cdot \frac{1}{7.8} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -6.8 \end{pmatrix}$$
$$= \frac{1}{7.8} \cdot \begin{pmatrix} 6.8 & 6.8 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 6.8/7.8 & 6.8/7.8 \\ 1/7.8 & 1/7.8 \end{pmatrix}$$
$$\simeq \underline{\begin{pmatrix} 0.872 & 0.872 \\ 0.128 & 0.128 \end{pmatrix}}$$

Conclusion: In the long run ($n \to \infty$), then state vector is

$$A^n \cdot \underline{v_0} \to \begin{pmatrix} 0.872 & 0.872 \\ 0.128 & 0.128 \end{pmatrix} \cdot \begin{pmatrix} 0.9 \\ 0.1 \end{pmatrix} = \underline{\begin{pmatrix} 0.872 \\ 0.128 \end{pmatrix}}$$

That is, unemployment is <u>12.8%</u>

<u>Explanation:</u> Why is $P^{-1}AP = D$ when

$$D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}, \quad P = (\underline{v_1} \mid \underline{v_2} \mid \dots \mid \underline{v_n})?$$

$$A \cdot P = A \cdot (\underline{v_1} \mid \underline{v_2} \mid \dots \mid \underline{v_n}) = (A\underline{v_1} \mid A\underline{v_2} \mid \dots \mid A\underline{v_n})$$

$$= (\lambda_1 \underline{v_1} \mid \lambda_2 \underline{v_2} \mid \dots \mid \lambda_n \underline{v_n})$$

$$P \cdot D = (\underline{v_1} \mid \underline{v_2} \mid \dots \mid \underline{v_n}) \cdot \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{pmatrix} = (\lambda_1 \underline{v_1} \mid \lambda_2 \underline{v_2} \mid \dots)$$

This means that $AP = PD$. If P is invertible, left multiplication with P^{-1} given

$$AP = PD$$

$$P^{-1}AP = P^{-1}PD = D$$

$$P^{-1}AP = D$$

Ex: $A = \begin{pmatrix} 7 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 7 \end{pmatrix}$

$$\begin{vmatrix} 7-\lambda & 0 & 3 \\ 0 & 2-\lambda & 0 \\ 3 & 0 & 7-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\cdot \begin{vmatrix} 7-\lambda & 3 \\ 3 & 7-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\cdot(\lambda^2 - 14\lambda + 40) = 0$$

$$\underline{\lambda_1 = 2} \quad \underline{\lambda_2 = 4}, \ \underline{\lambda_3 = 10}$$

choose (random) order of eigenvalues $\longrightarrow$ $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 10 \end{pmatrix}$

$\underline{\lambda_1 = 2}$:

$$\begin{pmatrix} 5 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} 5x + 3z &= 0 \\ 3x + 5x &= 0 \end{aligned} \quad \cdot(-3/5)$$

$$\begin{aligned} 5x + 3z &= 0 \\ \tfrac{16}{5} z &= 0 \end{aligned} \qquad \begin{aligned} x &= 0 \\ z &= 0 \\ y &= \text{free} \end{aligned}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} = \underline{y \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$$

$$\underline{v_1} = \underline{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$$

$\underline{\lambda_2 = 4}$:

$$\begin{pmatrix} 3 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & 3 \end{pmatrix} \qquad \begin{aligned} 3x + 3z &= 0 \\ -2y &= 0 \end{aligned} \qquad \begin{aligned} x &= -z \\ y &= 0 \\ z &= \text{free} \end{aligned}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ 0 \\ z \end{pmatrix} = z \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \underline{v_2} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$\underline{\lambda_3 = 10}$:

$$\begin{pmatrix} -3 & 0 & 3 \\ 0 & -8 & 0 \\ 3 & 0 & -3 \end{pmatrix} \qquad \begin{aligned} -3x + 3z &= 0 \\ -8y &= 0 \end{aligned} \qquad \begin{aligned} x &= z \\ y &= 0 \\ z &= \text{free} \end{aligned}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ 0 \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \underline{v_3} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

choose corresponding order of $\underline{v_1}, \underline{v_2}, \underline{v_3}$

$$P = \left(\underline{v_1} \mid \underline{v_2} \mid \underline{v_3} \right) = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 10 \end{pmatrix}$$

This means:

$$P^{-1} \cdot A \cdot P = D$$

Know this from theory, not necessary to multiply $P^{-1}AP$ to verify

P is invertible since eigenvectors are lin. independent by construction