LECTURE 13 (II)

GRA 6035 MATHEMATICS

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Plan:

(3) Differential / difference equations

(4) Constrained optimization problems

Exam problems:

Final 13/2013: Q.3–4

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(3) Differential / difference equations

Differential equations:

a) First order

$$y' = F(y,t)$$

Seperable:

$$y' = f(y) \cdot g(t)$$

$$\boxed{\int \frac{1}{f(y)}\,dy = \int g(t)\,dt}$$

Linear:

$$y' = -ay + b$$

$$y' = -a(t)y + b(t)$$

Use integrating factor $e^{\int a(t)dt}$

or (if a is a constant)

$$y = y_h + y_p = Ce^{-at} + y_p$$

b) Second order

$$y'' = F(y', y, t)$$

Linear: (with const. coeff.)

$$y'' + ay' + by = f(t)$$

$y = y_h + y_p$ and we use characteristic equation

$$r^2 + ar + b = 0$$

to find y_h

Exact first order:

$$p(y,t) + q(y,t) \cdot y' = 0$$

Solution: $h(y,t) = C$

where

$h'_t = p$

$h'_y = q$

Difference equations

a) **First order linear** (with const. coeffs.)

$$y_{t+1} = F(y_t, t)$$

$$y_{t+1} + a y_t = f_t$$

$$y_t = y_t^h + y_t^p = C \cdot (-a)^t + y_t^p$$

b) **Second order linear** (with const. coeff.)

$$y_{t+2} = F(y_{t+1}, y_t, t)$$

$$y_{t+2} + a y_{t+1} + b y_t = f_t$$

$$y_t = y_t^h + y_t^p$$

and

y_t^h is given by the solution of the char. eqn.

$$r^2 + ar + b = 0$$

Stability:

What happens to $y(t)$ / y_t when $t \to \infty$?

$$\bar{y} = \lim_{t \to \infty} y(t) \quad \text{or} \quad \bar{y} = \lim_{t \to \infty} y_t$$

Exam 12/2013 Q3:

a) $y_{t+2} - 11y_{t+1} + 28y_t = 36t+18$

$$y_t = y_t^h + y_t^p = \underbrace{C_1 \cdot 4^t + C_2 \cdot 7^t}_{y_t^h} + \underbrace{2t+2}_{y_t^p}$$

y_t^h: $r^2 - 11r + 28 = 0$

$r_1 = 4$ or $r_2 = 7$ $\longrightarrow$ $y_t^h = C_1 \cdot 4^t + C_2 \cdot 7^t$

$$r = \frac{11 \pm \sqrt{11^2 - 4 \cdot 28}}{2}$$

$$= \frac{11 \pm 3}{2} = 7, 4$$

y_t^p: $f_t = 36t+18$

$f_{t+1} = \ldots$

$f_{t+2} = \ldots$

$$\begin{cases} y_t = At+B \\ y_{t+1} = A(t+1)+B = At+A+B \\ y_{t+2} = A(t+2)+B = At+2A+B \end{cases}$$

$$(At+2A+B) - 11 \cdot (At+A+B) + 28(At+B) = 36t+18$$

$$(\underset{\underset{36}{\parallel}}{18A})t + (\underbrace{-9A+18B}_{18}) = 36t+18$$

$A = 2$ $\qquad -18+18B = 18$ $\qquad \longrightarrow y_t^p = At+B = 2t+2$

$B = 2$

b) $y' = \underbrace{4y + te^t}$

$y' - 4y = te^t$ linear first order diff-eqn.

$y = y_h + y_p = \underline{\underline{C \cdot e^{4t} + \left(-\frac{1}{3}t - \frac{1}{9}\right)e^t}}$

$\underline{y_h}$: $y' - 4y = 0$

$r - 4 = 0$ $\quad \underline{r = 4}$ $\quad y_h = C \cdot e^{4t}$

$\underline{y_p}$: $y' - 4y = te^t$

$f(t) = te^t$

$f'(t) = 1 \cdot e^t + t \cdot e^t = (t+1)e^t$

$f''(t) = (t+2)e^t$

$(At + A + B)e^t - 4(At + B)e^t = te^t$

$y = (At + B)e^t$

$y' = A \cdot e^t + (At + B) \cdot e^t$

$= (At + A + B)e^t$

$(-3A)te^t + (A - 3B)e^t = te^t$

$-3A = 1 \qquad A - 3B = 0$

$A = \underline{-1/3}$ $\qquad -\frac{1}{3} - 3B = 0$

$B = \underline{-1/9}$ $\quad \rightarrow y_p = (At + B)e^t$

$= \underline{\left(-\frac{1}{3}t - \frac{1}{9}\right)e^t}$

Alt: Integrating factor

$y' - 4y = te^t$

$a = -4$

$\int a \, dt = -4t \rightarrow u = \underline{e^{-4t}}$

$(y \cdot e^{-4t})' = te^t \cdot e^{-4t}$

$= t \cdot e^{-3t}$

$y \cdot e^{-4t} = \int te^{-3t} \, dt$

c) $$\underbrace{\frac{y}{y^2+t^2+3}}_{q} \cdot y' + \underbrace{\frac{t}{y^2+t^2+3}}_{P} = 0 \quad , \; y(1) = 2$$

Exact:

$$h'_t = P = \frac{t}{y^2+t^2+3}$$

$$h'_y = q = \frac{y}{y^2+t^2+3}$$

$$h = \int \frac{t}{y^2+t^2+3}\,dt$$

$$\boxed{\begin{array}{l} u = y^2+t^2+3 \\ du = 2t\,dt \end{array}}$$

$$= \int \frac{\cancel{t}}{u} \cdot \frac{du}{2\cancel{t}}$$

$$= \frac{1}{2}\int \frac{1}{u}\,du$$

$$= \frac{1}{2}\ln(y^2+t^2+3) + C(y)$$

$$h'_y = \frac{y}{y^2+t^2+3} + C'(y)$$

$$= \frac{y}{y^2+t^2+3}$$

Can put $C(y) = 0$: $h = \frac{1}{2}\ln(y^2+t^2+3)$

Solution: $h = \frac{1}{2}\ln(y^2+t^2+3) = C$

$$\ln(y^2+t^2+3) = 2C$$

$$y^2+t^2+3 = e^{2C}$$

$$y^2 = -t^2-3+e^{2C} = -t^2-3+8 = \underline{\underline{-t^2+5}}$$

$y(1) = 2$

$$4 = 1 - 3 + e^{2C}$$

$$8 = e^{2C}$$

$$y = \underline{\underline{\sqrt{5-t^2}}}$$

Alt: $1 \cdot (y^2+t^2+3)$

$$yy' + t = 0$$

$$yy' = -t$$ Seperable

$$y' = -\frac{t}{y}$$

$$\int yy'\,dt = \int -t\,dt$$

$$\frac{1}{2}y^2 = -\frac{1}{2}t^2 + C$$

$$\underline{y^2 = -t^2 + 2C}$$

$$2^2 = -1^2 + 2C$$

$$5 = 2C$$

$$y^2 = -t^2 + 5$$

$$y = \underline{\underline{\sqrt{5-t^2}}}$$

4 Constrained optimization:

a) Lagrange problem:

$$\max/\min f(\underline{x}) \quad \text{when} \quad \begin{cases} g_1(\underline{x}) = a_1 \\ g_2(\underline{x}) = a_2 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases}$$

$\underline{x} = x_1, \ldots, x_n$

b) Kuhn-Tucker problem

$$\max f(\underline{x}) \quad \text{when} \quad \begin{cases} g_1(\underline{x}) \leq a_1 \\ \vdots \\ g_m(\underline{x}) \leq a_m \end{cases}$$

(Std. form)

i) Candidates for optimum:

FOC: first order conditions

$$\mathcal{L} = f(\underline{x}) - \lambda_1 \cdot g_1(\underline{x}) - \lambda_2 \cdot g_2(\underline{x}) - \cdots - \lambda_m g_m(\underline{x})$$

$$\boxed{\text{FOC:} \quad \begin{aligned} \mathcal{L}'_{x_1} &= 0 \\ \mathcal{L}'_{x_2} &= 0 \\ &\vdots \\ \mathcal{L}'_{x_n} &= 0 \end{aligned}}$$

$$\begin{cases} g_1(\underline{x}) = a_1 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases}$$

a) Lagrange case:

Solve Lagrange conditions (FOC) + (C)

$$\begin{cases} g_1(\underline{x}) \leq a_1 \\ \vdots \\ g_m(\underline{x}) \leq a_m \end{cases}$$

b) Kuhn-Tucker case:

Solve Kuhn-Tucker conditions (FOC) + (C) + (CSC)

CSC:

$$\begin{cases} \lambda_1 \geq 0 & \text{and} \quad \lambda_1 \cdot (g_1(\underline{x}) - a_1) = 0 \\ \lambda_2 \geq 0 & \text{"} \quad \lambda_2 \cdot (g_2(\underline{x}) - a_2) = 0 \\ \vdots & \\ \lambda_m \geq 0 & \text{"} \quad \lambda_m (g_m(\underline{x}) - a_m) = 0 \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} \lambda_1 \geq 0, \text{ and } \lambda_1 = 0 & \text{if } g_1(\underline{x}) < a_1 \\ \lambda_2 \geq 0, \text{ and } \lambda_2 = 0 & \text{if } g_2(\underline{x}) < a_2 \\ \vdots & \\ \lambda_m \geq 0, \text{ and } \lambda_m = 0 & \text{if } g_m(\underline{x}) < a_m \end{cases}$$

Solve FOC+C / FOC+C+CSC to find candidates for max/min.

ii) Solve Lagrange / Kuhn-Tucker problem:

SOC: If $(\underline{x}^*; \underline{\lambda}^*)$ solves Lagrange / Kuhn-Tucker conditions, we consider $\mathcal{L}(x_1, \dots, x_n; \lambda_1^*, \lambda_2^*, \dots, \lambda_m^*)$:

concave function $\Rightarrow \underline{x}^*$ is max
convex " $\Rightarrow \underline{x}^*$ is min

EVT: (1) If the set of adm. pts (pts satisfying all constraints) is bounded, then there is always max/min.

(2) List of possible solutions:

- All pts that satisfy Lagrange / Kuhn-Tucker cond.
- Pts that fail NDCQ. (and are admissible)

iii) <u>Envelope thm</u> : a is a parameter

$$\boxed{\frac{df^*(a)}{da} = \frac{\partial h}{\partial a}\left(\underline{x}^*(a);\lambda^*(a)\right)}$$

where $(x^*(a);\lambda^*(a))$ is ~~the~~ solution of the Lagrange / Kuhn-Tucker problem for parameter a, and $(x^*(a);\lambda^*(a))$ satisfy FOC+C / FOC + C+CSC.

<u>Note</u>: You need to write the constraints in the form

$$g(\underline{x}) = 0 \quad / \quad g(x) \leq 0$$

(move the constant a into the fn g) to use the envelope thm. for Lagrange problems / Kuhn-Tucker problems

Exam 2013/12 Q4.

$$\max \underbrace{xw - yz}_{f} \quad \text{when} \begin{cases} \overbrace{x^2+y^2}^{g_1} \le 1 = a_1 \\ \underbrace{4z^2+9w^2}_{g_2} \le 36 = a_2 \end{cases}$$ (std. form)

a) $\mathcal{L} = xw - yz - \lambda_1 \cdot (x^2+y^2) - \lambda_2 \cdot (4z^2+9w^2)$

$\mathcal{L}'_x = w - \lambda_1 \cdot 2x$

$\mathcal{L}'_y = -z - \lambda_1 \cdot 2y$

$\mathcal{L}'_z = -y - \lambda_2 \cdot 8z$

$\mathcal{L}'_w = x - \lambda_2 \cdot 18w$

Kuhn-Tucker conditions: FOC + C + CSC

FOC: $w - 2\lambda_1 x = 0$, $-z - 2\lambda_1 y = 0$, $-y - 8\lambda_2 z = 0$, $x - 18\lambda_2 w = 0$

C: $x^2+y^2 \le 1$, $4z^2+9w^2 \le 36$

CSC: $\lambda_1 \ge 0$, $\lambda_2 \ge 0$, $\lambda_1 \cdot (x^2+y^2-1) = 0$, $\lambda_2 \cdot (4z^2+9w^2-36) = 0$

$(x,y,z,w) = (0,1,-3,0)$.

FOC: $0=0$ ✓
$3 - 2\lambda_1 = 0 \quad \lambda_1 = 3/2$
$-1 + 24\lambda_2 = 0 \quad \lambda_2 = 1/24$
$0=0$ ✓

C: $0^2+1^2 = 1 \le 1$ ✓
$4 \cdot 9 + 0 = 36 \le 36$ ✓

CSC: $\lambda_1 \ge 0$ ✓
$\lambda_2 \ge 0$ ✓
$\lambda_1 \cdot 0 = 0$ ✓
$\lambda_2 \cdot 0 = 0$ ✓

⇓

One solution to KT-cond: $(0,1,-3,0; \frac{3}{2}, \frac{1}{24})$ $f = 3$

b) $(x,y,z,w;\lambda_1,\lambda_2) = (0,+1,-3,0;\ \frac{3}{2},\frac{1}{24})$ $\quad f=3$

$$\mathcal{L} = xw - yz - \lambda_1(x^2+y^2) - \lambda_2(4z^2+9w^2)$$
$$= xw - yz - \frac{3}{2}(x^2+y^2) - \frac{1}{24}(4z^2+9w^2) = h$$

SOC: this function is concave $\Longrightarrow$ $(0,1,-3,0)$ is max. point

$h'_x = w - 3x$

$h'_y = -z - 3y$

$h'_z = -y - \frac{1}{3}z$

$h'_w = x - \frac{3}{4}w$

$$H(h) = \begin{pmatrix} -3 & 0 & 0 & 1 \\ 0 & -3 & -1 & 0 \\ 0 & -1 & -1/3 & 0 \\ 1 & 0 & 0 & -3/4 \end{pmatrix}$$

$\Delta_1 = -3, -3, -\frac{1}{3}, -3/4 \leq 0$

$\Delta_2 = 9, 1, 5/4, 0, \frac{9}{4}, \frac{1}{4} \geq 0$

$\Delta_3 = \overset{123}{0}, \underset{124}{\cancel{\quad}}, \overset{134}{\cancel{\quad}}, \overset{234}{0} \leq 0$

$\Delta_4 = 0 \geq 0$

$$\Delta_{124} = \begin{vmatrix} -3 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & -3/4 \end{vmatrix} = -3 \cdot 5/4 = -15/4 < 0$$

$$\Delta_{134} = \begin{vmatrix} -3 & 0 & 1 \\ 0 & -1/3 & 0 \\ 1 & 0 & -3/4 \end{vmatrix} = -\frac{1}{3}\cdot\frac{5}{4} = -\frac{5}{12} < 0$$

$D_1 = -3 < 0$

$D_2 = 9 > 0$

$D_3 = -3 \cdot (1-1) = 0$

$D_4 = -3/4 \cdot \underset{0}{D_3} - 1 \cdot 1 \cdot 0 = 0$

$$\begin{pmatrix} -3 & 0 & 0 & 1 \\ 0 & -3 & -1 & 0 \\ 0 & -1 & -1/3 & 0 \\ 1 & 0 & 0 & -3/4 \end{pmatrix} \begin{matrix} \\ \frac{1}{3} \\ -1/3 \\ \\ \end{matrix}$$

$\downarrow$

$$\begin{pmatrix} -3 & 0 & 0 & 1 \\ 0 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * \end{pmatrix}$$

not rank 2

Concl: $f(0,1,-3,0)=3$ is max since h is concave

c) $\max \; xw - yz$ when $\begin{cases} x^2+y^2 \le 1 \\ 4.2z^2+9w^2 \le 36 \end{cases}$ changed from $c=4$ to 4.2

The parameter c has changed from $c=4$ to $c=4.2$:

$$\boxed{\max \; xw - yz \quad \text{when} \begin{cases} x^2+y^2 \le 1 \\ cz^2+9w^2 \le 36 \end{cases}}$$

$x^2+y^2-1 \le 0$

$cz^2+9w^2-36 \le 0$

Env. thm: $$\frac{df^*(c)}{dc} = \frac{\partial L}{\partial c}\left(0,1,-3,0;\tfrac{3}{2},\tfrac{1}{24}\right)$$

$$L = xw - yz - \lambda_1 \cdot (x^2+y^2-1) - \lambda_2(cz^2+9w^2-36)$$

$$\frac{\partial L}{\partial c} = -\lambda_2 z^2 \qquad \frac{\partial L}{\partial c}\left(0,1,-3,0;\tfrac{3}{2},\tfrac{1}{24}\right) = -\frac{1}{24}\cdot(-3)^2 = -\frac{9}{24}$$

Estimate the new max value when $c=4.2$

$$\frac{df^*(c)}{dc} = -\frac{9}{24} \Rightarrow f^*(4.2) \simeq f^*(4) + 0.2\cdot\left(-\frac{9}{24}\right)$$
$$= 3 - 0.2\cdot\frac{9}{24}$$
$$= 3 - 0.075$$
$$= \underline{\underline{2.925}}$$

$\Delta c = 4.2-4 = 0.2$

d) FOC:

$$\boxed{\begin{aligned} w - 2\lambda_1 x &= 0 \\ -z - 2\lambda_1 y &= 0 \\ -y - 8\lambda_2 z &= 0 \\ x - 18\lambda_2 w &= 0 \end{aligned}}$$

C: $x^2+y^2 \leq 1$
$4z^2+9w^2 \leq 36$

CSC: $\lambda_1 \geq 0$
$\lambda_2 \geq 0$

$w = 2\lambda_1 x$
$x = 18\lambda_2 w$

$x = 18\lambda_2 \cdot 2\lambda_1 x$
$x = 36\lambda_1\lambda_2 x$

$\boxed{\begin{aligned} x&=0 \\ w&=0 \end{aligned}}$ or $\boxed{\begin{aligned} 1&=36\lambda_1\lambda_2 \\ w&=2\lambda_1 x \end{aligned}}$

$z = -2\lambda_1 y$
$y = -8\lambda_2 z$

$y = -8\lambda_2 \cdot (-2\lambda_1 y)$
$= 16\lambda_1\lambda_2 y$

$\boxed{\begin{aligned} y&=0 \\ z&=0 \end{aligned}}$ or $\boxed{\begin{aligned} 1&=16\lambda_1\lambda_2 \\ y&=-8\lambda_2 z \end{aligned}}$

1) $x=0, w=0$ and $y=0, z=0$ $\Rightarrow$ $\boxed{(0,0,0,0) \quad \lambda_1=\lambda_2=0 \quad f=0}$

2) $x=0, w=0$ and $\lambda_1\lambda_2 = 1/16$ $\Rightarrow$ $x=0, w=0$, $\lambda_1, \lambda_2 > 0$

CSC: $x^2+y^2=1$ $\quad y=\pm 1$
$4z^2+9w^2=36$ $\quad z=\pm 3$

$$\boxed{\begin{aligned} &(0,1,-3,0;\ \tfrac{3}{2}, \tfrac{1}{24}) \quad f=3 \\ &(0,-1,3,0;\ \tfrac{3}{2}, \tfrac{1}{24}) \quad f=3 \end{aligned}}$$

$-1 = -8\lambda_2 \cdot 3$

$\lambda_2 = 1/24$ $\quad \lambda_1\lambda_2 = \frac{1}{16}$

$\lambda_1 \frac{1}{24} = \frac{1}{16}$

$\lambda_1 = \frac{24}{16} = \frac{3}{2}$

3) $\lambda_1\lambda_2 = \frac{1}{36}$ and $y=z=0$

$\boxed{\text{Same way as 2)}}$

$y=0, z=0$, $\lambda_1, \lambda_2 > 0$

CSC: $x^2+y^2=1$ $\quad x=\pm 1$
$4z^2+9w^2=36$ $\quad w=\pm 2$

$$\boxed{\begin{aligned} &(1,0,0,2;\ 1, 1/36) \quad f=2 \\ &(-1,0,0,-2;\ 1; 1/36) \quad f=2 \end{aligned}}$$

$w = 2\lambda_1 x$
$\Rightarrow \lambda_1 = 1$
$\lambda_2 = 1/36$