LECTURE 12 (F)

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GRA 6035 MATHEMATICS

Plan:

1. Difference equations
   a) First order linear
   b) Second order linear
2. Stability
   a) Difference equations
   b) Differential equations

Reading:

[ME] 23.2

Next week: Lecture Monday (at 1700) + Thursday (at 0800)

Revision + Exam Dec. '13

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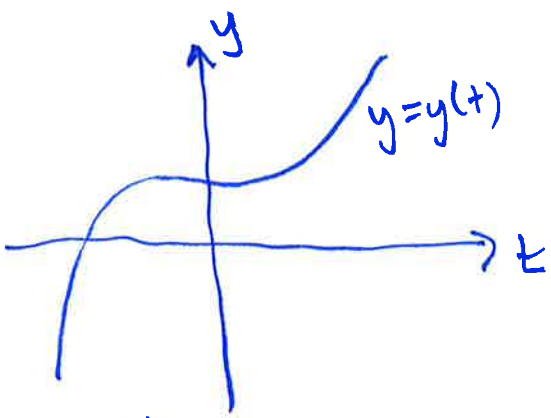
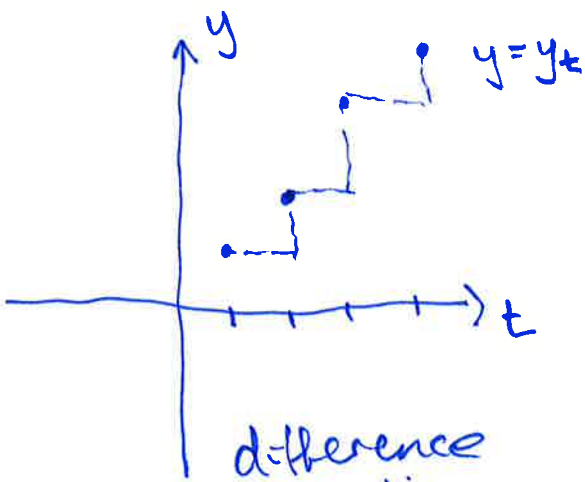
1 Difference equations:

An equation relating terms in a sequence (y_t) to each other.

Ex: Markow chains $\underline{x}_{t+1} = A \cdot \underline{x}_t$



differential eqn.



difference equation

Ex: Bank account, interest r

$$\boxed{B_t = B_{t-1} + r \cdot B_{t-1}}$$

difference equation = recurrence relation

$$B_t = (1+r)B_{t-1}$$

B_0 $\quad B_1 = B_0 \cdot (1+r) \quad B_2 = B_1 \cdot (1+r) \quad \ldots\ldots$

$$= B_0 (1+r)^2$$

↑ starting value of bank account

closed form solution:

$$B_t = B_0 \cdot (1+r)^t$$

Differential equation:

$$B' = r \cdot B$$

$$\frac{1}{B} \cdot B' = r \rightsquigarrow B(t) = B_0 \cdot e^{rt}$$

Note:

$$\left.\begin{aligned} B_t - B_{t-1} &= r \cdot B_{t-1} \\ B' &= r \cdot B \end{aligned}\right\}$$

both differential and difference eqn's are used to model change

Ex: $y_{t+1} = 1.04\, y_t \,,\; y_0 = 100$

$y_0 = 100 \qquad y_1 = 1.04 \cdot 100 = 104 \qquad y_2 = 1{,}04 \cdot y_1 = 1{,}04 \cdot 104 = 1{,}04^2 \cdot 100$

Solution: $\underline{\underline{y_t = 100 \cdot 1{,}04^t}}$

② Linear difference equations

A difference equation is called first order if it can be written

$$y_{t+1} = F(t, y_t)$$

$$\Updownarrow$$

$$\left(y_{t+1} - y_t = F(t, y_t) - y_t \right)$$

It is called linear if it can be written

$$y_{t+1} + a y_t = b_t \qquad \text{(a is a constant)}$$

i) Homogeneous case: $b_t = 0$

$$y_{t+1} + a y_t = 0$$

$$y_{t+1} = -a \cdot y_t$$

$y_1 = -a \cdot y_0 \qquad y_2 = -a \cdot y_1 = (-a)^2 y_0 \quad \ldots \quad \boxed{y_t = (-a)^t \cdot y_0}$

Characteristic equations:

$y_{t+1} + a y_t = 0 \rightsquigarrow$ Characteristic eqn:

$$r + a = 0$$

Solution: $\Longleftarrow$ $\underline{r = -a}$

$$y_t = C \cdot (\underbrace{-a}_{= r})^t$$

Why does this work?

Guess solution: $\begin{cases} y_t = r^t \\ y_{t+1} = r^{t+1} = r^t \cdot r \end{cases}$

$$y_{t+1} + a y_t = 0$$

$$r \cdot r^t + a \cdot r^t = 0$$

$$r^t \cdot (r + a) = 0$$

$$\underline{r + a = 0}$$

ii) Inhomogeneous case: $y_{t+1} + a y_t = b_t$

Superposition principle:

General solution is $y_t = y_t^h + y_t^p$, where

y_t^h: general solution of homog. eqn.: $y_{t+1} + a y_t = 0$

y_t^p: particular solution of $y_{t+1} + a y_t = b_t$

Ex:

$$y_{t+1} - 1.05\, y_t = -100 , \quad y_0 = 1000$$

$$\left(y_{t+1} - y_t = 0.05\, y_t - 100 , \quad y_0 = 1000 \right)$$

$$y_t = y_t^h + y_t^p = C \cdot 1.05^t + 2000$$

y_t^h:

$$y_{t+1} - 1.05\, y_t = 0$$

$$r - 1.05 = 0$$

$$r = 1.05 \longrightarrow y_t^h = C \cdot 1.05^t$$

y_t^p:

$$y_{t+1} - 1.05\, y_t = -100$$

Guess: $y_t = A$

$y_{t+1} = A$

$$A - 1.05\, A = -100$$

$$-0.05\, A = -100$$

$$A = \frac{-100}{-0.05} = 2000$$

$$y_t^p = 2000$$

$y_0 = 1000$:

$$y_t = C \cdot 1.05^t + 2000$$

$$1000 = C \cdot 1.05^0 + 2000$$

$$C = 1000 - 2000 = -1000$$

$$y_t = 2000 - 1000 \cdot 1.05^t$$

Summary: First order linear difference equations (with constant coeff.)

i) Homogeneous case: $y_{t+1} + a y_t = 0$

$$y_t = \underline{C \cdot (-a)^t}$$

ii) Inhomogeneous case: $y_{t+1} + a y_t = b_t$

$$y_t = \underline{C \cdot (-a)^t + y_t^P}$$

Second order linear difference equations with constant coefficients

$$\Delta y_t = y_t - y_{t-1}$$

$$\Delta^2 y_t = (y_t - y_{t-1}) - (y_{t-1} - y_{t-2})$$

$$= y_t - 2y_{t-1} + y_{t-2}$$

"change in the change"

An equation that includes y_{t+2}, y_{t+1}, y_t is a second order difference equation.

i) Homogeneous case: $y_{t+2} + a y_{t+1} + b y_t = 0$

Guess: $y_t = r^t$

$y_{t+1} = r^{t+1} = r \cdot r^t$

$y_{t+2} = r^{t+2} = r^2 \cdot r^t$

$\Big\}$ $r^2 \cdot r^t + a(r \cdot r^t) + b r^t = 0$

$r^t \cdot (r^2 + ar + b) = 0$

Char. eqn: $\longrightarrow$ $r^2 + ar + b = 0$

$$r = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

i) Two roots $r_1 \neq r_2$: $y_t = C_1 \cdot r_1^t + C_2 \cdot r_2^t$

ii) One ~~double~~ root r_1: $y_t = C_1 \cdot r_1^t + C_2 t \cdot r_1^t$

$= (C_1 + C_2 t) r_1^t$

iii) No real roots: $y_t = (\sqrt{b})^t \cdot (C_1 \cdot \cos \theta t + C_2 \cdot \sin \theta t)$

where $\theta = \cos^{-1}\left(-\frac{a}{2\sqrt{b}}\right)$

Ex: $y_{t+2} = y_{t+1} + y_t$, $y_0 = 1$, $y_1 = 1$

$y_{t+2} - y_{t+1} - y_t = 0$

$y_0 = 1, y_1 = 1, y_2 = 2$

$y_3 = 3, y_4 = 5,$

$y_6 = 8, y_7 = 13,$

...

Char. eqn: $r^2 - r - 1 = 0$

$$r = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$y_t = C_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^t + C_2 \cdot \left(\frac{1-\sqrt{5}}{2}\right)^t$$

$$y_t = c_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^t + c_2 \cdot \left(\frac{1-\sqrt{5}}{2}\right)^t$$

$$y_0 = 1 \quad : \quad 1 = c_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^0 + c_2 \cdot \left(\frac{1-\sqrt{5}}{2}\right)^0 = c_1 + c_2$$

$$y_1 = 1 \quad : \quad 1 = c_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^1 + c_2 \cdot \left(\frac{1-\sqrt{5}}{2}\right)^1$$

$$2 = c_1 \cdot (1+\sqrt{5}) + (1-c_1) \cdot (1-\sqrt{5})$$

$$(2-1+\sqrt{5}) = c_1 \cdot (\cancel{1}+\sqrt{5} \cancel{-1} + \sqrt{5})$$

$$c_1 = \frac{1+\sqrt{5}}{2\sqrt{5}} \qquad c_2 = 1 - \frac{1+\sqrt{5}}{2\sqrt{5}} = \frac{2\sqrt{5}-(1+\sqrt{5})}{2\sqrt{5}} = \frac{\sqrt{5}-1}{2\sqrt{5}}$$

$$y_t = \frac{\sqrt{5}+1}{2\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^t + \frac{\sqrt{5}-1}{2\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^t$$

Ex: $y_{t+2} - 4y_{t+1} + 4y_t = 0$

$$r^2 - 4r + 4 = 0$$

$r = 2$ double root $\rightarrow y_t = c_1 \cdot 2^t + c_2 \cdot t \cdot 2^t$

$$= (c_1 + c_2 t) \cdot 2^t$$

Inhomogeneous case: $\boxed{y_{t+2} + a y_{t+1} + b y_t = f_t}$

General solution: $y_t = y_t^h + y_t^p$

general homogeneous solution $\left(y_{t+2} + a y_{t+1} + b y_t = 0 \right)$

particular solution $\left(y_{t+2} + a y_{t+1} + b y_t = f_t \right)$

Ex: $y_{t+2} - y_{t+1} - 2 y_t = 3^t$

General solution: $y_t = y_t^h + y_t^p$

$= C_1 \cdot 2^t + C_2 \cdot (-1)^t$

y_t^h: $y_{t+2} - y_{t+1} - 2 y_t = 0$

$r^2 - r - 2 = 0$

$$r = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2}$$

$r_1 = 2, \quad r_2 = -1 \longrightarrow y_t^h = C_1 \cdot 2^t + C_2 \cdot (-1)^t$

y_t^p: $y_{t+2} - y_{t+1} - 2 y_t = 3^t$

$f_t = 3^t$

$f_{t+1} = 3^{t+1} = 3 \cdot 3^t$

$f_{t+2} = 3^{t+2} = 9 \cdot 3^t$

Guess: $y_t = A \cdot 3^t$

$y_{t+1} = 3A \cdot 3^t$

$y_{t+2} = 9A \cdot 3^t$

make a guess of the same form as f_t and with a parameter

it is useful to compute f_{t+1}, f_{t+2} to make a good guess

$$y_{t+2} - y_{t+1} - 2y_t = 3^t$$

$$9A \cdot 3^t - 3A \cdot 3^t - 2 \cdot A \cdot 3^t = 3^t$$

$$3^t \cdot (9A - 3A - 2A) = 3^t$$

$$+4A = 1$$

$$A = +1/4 \Rightarrow y_t^P = \underline{+\tfrac{1}{4} \cdot 3^t}$$

$$y_t = \underbrace{C_1 \cdot 2^t + C_2 \cdot (-1)^t}_{y_t^h} + \underbrace{\tfrac{1}{4} \cdot 3^t}_{y_t^P}$$

If the initial guess does not work, try to multiply it with t.

Ex: $y_{t+2} - 2y_{t+1} + y_t = -15$

$$y_t = y_t^h + y_t^p = \underline{\underline{c_1 + c_2 t - \tfrac{15}{2} t^2}}$$

$\underline{y_t^h}$: $y_{t+2} - 2y_{t+1} + y_t = 0$

$$r^2 - 2r + 1 = 0$$

$$(r-1)^2 = 0$$

$r = 1$ double root $\rightarrow$ $y_t^h = c_1 \cdot 1^t + c_2 t \cdot 1^t$

$$= \underline{c_1 + c_2 t}$$

$\underline{y_t^p}$: $y_{t+2} - 2y_{t+1} + y_t = -15$

$f_t = -15$

$f_{t+1} = -15$

$f_{t+2} = -15$

$\shortparallel$

$A - 2A + A = -15 \leftarrow$ Guess: $y_t = A$

$0 \cdot A = -15$

<u>no solution</u>

<u>Guess</u>: $y_t = At$

$y_{t+1} = At + A$

$y_{t+2} = At + 2A$

$(At + 2A) - 2(At + A) \leftarrow$

$+ At = -15$

$0 \cdot t + 0 = -15$

no solution.

Guess: $y_t = At^2$

$y_{t+1} = A \cdot (t+1)^2$

$y_{t+2} = A \cdot (t+2)^2$

$A \cdot (t^2 + 4t + 4) - 2 \cdot A \cdot (t^2 + 2t + 1) \leftarrow$

$+ At^2 = -15$

$4A - 2A = -15$

$A = -15/2 \longrightarrow y_t^p = -\frac{15}{2} t^2$

② Stability:

Differential equation $\longrightarrow$ $y = y(t)$ solution

Difference equation $\longrightarrow$ $y = y_t$ solution

It is called <u>globally asymptotically stable</u>

if

$$\bar{y} = \lim_{t\to\infty} y = \begin{cases} \lim_{t\to\infty} y(t) \\ \lim_{t\to\infty} y_t \end{cases}$$

satisfy i) $\bar{y}$ is finite ii) $\bar{y}$ does not depend on the undetermined coefficients

<u>Interpretation:</u> There is a long-run equilibrium $\bar{y}$, which is finite and independent of initial conditions.

<u>Ex:</u> $y_{t+2} - y_{t+1} - 2y_t = 3^t$

$y_t = C_1 \cdot 2^t + C_2 \cdot (-1)^t + \frac{1}{4} \cdot 3^t \longrightarrow \infty$ (as $t \to \infty$)

<u>not stable</u> (unstable)

<u>Ex:</u> $y'' - 3y' + 2y = 4$

$y = y_h + y_p = \underline{C_1 \cdot e^t + C_2 \cdot e^{2t} + 2} \to \pm\infty$ (unless $C_1 = C_2 = 0$)

y_h: $r^2 - 3r + 2 = 0$

$r = 1, r = 2$

y_p: $y = A = 2$

<u>not globally asymptotically stable</u>

For globally asymptotical stability, it is necessary that "essentially"

$|r| < 1$ for difference eqn's
$r < 0$ " differential equations

Ex: Systems of difference equations

$$\boxed{\begin{aligned} x_{t+1} &= 2x_t - y_t \\ y_{t+1} &= x_t \end{aligned}}$$

similar examples are Markov chains.

$$x_{t+1} = 2x_t - x_{t-1}$$

$$x_{t+1} - 2x_t + x_{t-1} = 0$$

$$r^2 - 2r + 1 = 0$$

$$r = 1 \longrightarrow x_t = (c_1 + c_2 t)\cdot 1^t = C_1 + C_2 t$$

$$y_t = x_{t-1} = c_1 + c_2(t-1)$$

$$= C_1 + C_2 t - c_2$$