LECTURE 12 (B)

GRA 6035 MATHEMATICS

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Plan:

1. Difference equations
2. Stability

Reading:

[ME] 23.2

Next week:

Lectures { Mon. 17.00, Th 08.00 }

Revision + Final exam 12/12 2013

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1. Difference equations

Ex: Markov chains

$$\underline{x}_{t+1} = A \cdot \underline{x}_t$$

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} 0.7 & 0.25 \\ 0.3 & 0.75 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

$$x_{t+1} = 0.7x_t + 0.25y_t$$
$$y_{t+1} = 0.3x_t + 0.75y_t$$

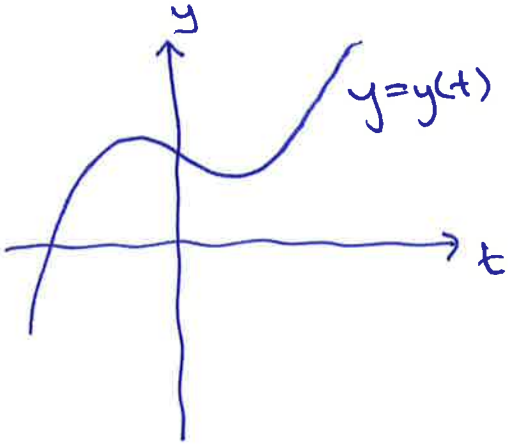
Ex:

$$y_{t+1} = 1.05y_t - 100$$

Solution:

Sequence of number $y_0, y_1, y_2, \ldots$

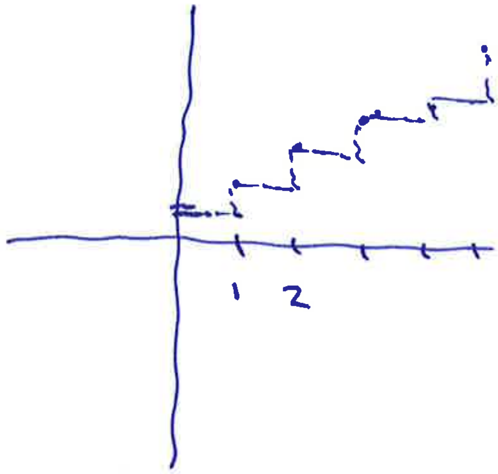
Defn: A *difference equation* is an equation that relates terms in a sequence with earlier terms. Also called *recurrence relations*.



differential equation

$$y' = F(y,t)$$

<u>change</u>: y'



difference equation

$$y_{t+1} = F(y_t, t)$$

<u>change</u>: $y_{t+1} - y_t$

$$y_{t+1} - y_t = F(y_t, t) - y_t$$

<u>Ex</u>: Bank account, interest r

$$B_{t+1} - B_t = r \cdot B_t$$
$$B_{t+1} = B_t + rB_t$$
$$B_{t+1} = (1+r) \cdot B_t$$

$$B_1 = (1+r) B_0$$
$$B_2 = (1+r) B_1$$
$$= (1+r)^2 \cdot B_0$$
$$\vdots$$
$$\underline{B_t = (1+r)^t \cdot B_0}$$

(closed formula)

<u>Ex</u>: $y_{t+1} = 1{,}04 \cdot y_t$

$$\Downarrow$$

$$\underline{y_t = y_0 \cdot 1{,}04^t}$$

(2) Difference equations : Linear case

A difference equation has first order if it relates y_{t+1} to y_t, i.e.

$$y_{t+1} = F(y_t, t)$$

It is called first order linear if it has the form (with constant coeffs.)

$$\boxed{y_{t+1} + a \cdot y_t = b_t}$$

where a is a constant, b_t is an expression in t.

Superposition principle:

The general solution $y = y_t$ of a linear difference equation is

$$y_t = y_t^h + y_t^p$$

where y_t^h is the general solution of the homogeneous equation, and y_t^p is a particular solution of the original equation.

i) Homogeneous case:

$$y_{t+1} + a y_t = 0$$

$$y_{t+1} = (-a) \cdot y_t$$

$$y_1 = (-a) y_0$$

$$y_2 = (-a) \cdot y_1 = (-a)^2 y_0$$

$$y_3 = (-a) y_2 = (-a)^3 y_0$$

$$\vdots$$

$$y_t = (-a)^t \cdot y_0$$

The solution in the homogeneous case

$y_{t+1} + a y_t = 0 \Rightarrow \quad \underline{y_t = C \cdot (-a)^t}$

Ex: $y_{t+1} - 1.04 y_t = 0 \quad , y_0 = 100$

$y_{t+1} = 1.04 y_t \longrightarrow \underline{y_t = C \cdot 1.04^t}$

$\underline{y_0 = 100}$: $\left.\begin{array}{l} t=0,\ y_t = 100 \\ 100 = C \cdot 1.04^0 \\ \underline{C = 100} \end{array}\right\}$ $\quad \underline{\underline{y_t = 100 \cdot 1.04^t}}$

(closed form)

ii) $\underline{\text{Inhomogeneous case:}}$ $\quad y_{t+1} + a y_t = b_t$

Ex: $y_{t+1} = 1.05 y_t - 100 \quad , \quad y_0 = 1000$

$y_{t+1} - 1.05 y_t = -100 \qquad \begin{cases} a = -1.05 \\ b_t = -100 \end{cases}$

$$y_t = y_t^h + y_t^p = \underline{C \cdot 1.05^t + 2000}$$

$\underline{y_t^h}$: $\quad y_{t+1} - 1.05 y_t = 0$

$y_t = (1.05)^t \cdot C \longrightarrow y_t^h = C \cdot 1.05^t$

$\underline{y_t^p}$: $\quad y_{t+1} - 1.05 y_t = -100 \qquad \underline{\text{Guess}}$: $y_t = A$

$A - 1.05 \cdot A = -100 \qquad y_{t+1} = A$

$-0.05A = -100$

$A = \frac{-100}{-0.05} = \underline{2000} \rightarrow \underline{y_t^p = 2000}$

$\underline{y_0 = 1000}$: $\quad y_t = C \cdot 1.05^t + 2000 = \underline{\underline{2000 - 1000 \cdot 1.05^t}}$

$1000 = C \cdot 1.05^0 + 2000$

$1000 = C + 2000 \Rightarrow C = \underline{-1000}$

Summary: First order linear difference equations with const. coeffs.

i) Homogeneous case:

$$y_{t+1} + a y_t = 0 \longrightarrow y_t = C \cdot (-a)^t$$

Char. eqn: $r + a = 0$
$r = -a$

ii) Inhomogeneous case:

$$y_{t+1} + a y_t = b_t \longrightarrow y_t = C \cdot (-a)^t + y_t^P$$

Second order linear difference equation with const. coeffs.

$$y_{t+2} + a y_{t+1} + b y_t = f_t$$

a, b : constants
f_t : an expression in t

Ex: Fibonacci

$$y_{t+2} = y_{t+1} + y_t, \quad \underbrace{y_0 = 1, y_1 = 1}_{\text{initial values}}$$

$$y_{t+2} - y_{t+1} - y_t = 0$$

$$\left.\begin{aligned} y_2 &= y_1 + y_0 = 2 \\ y_3 &= y_2 + y_1 = 3 \\ y_4 &= y_3 + y_2 = 5 \\ y_5 &= 8 \\ y_6 &= 13 \\ y_7 &= 21 \\ &\vdots \end{aligned}\right\} y_t = ?$$

A second order difference equation has the form

$$y_{t+2} = F(y_{t+1}, y_t, t)$$

y' : change in y

y'' : change in y'

= change in change

differential equations

$\Delta y_t =$

$y_{t+1} - y_t$: change in y

$$\cancel{\Delta^2 y_t \neq \Delta y_t - \Delta y_{t+1} = (y_{t+1} - y_t) - (y_{t+2}}$$

$$\Delta^2 y_t = \Delta y_{t+1} - \Delta y_t$$

$$= (y_{t+2} - y_{t+1}) - (y_{t+1} - y_t)$$

$$= \underline{y_{t+2} - 2y_{t+1} + y_t}$$

i) Homogeneous case: $y_{t+2} + a \cdot y_{t+1} + b y_t = 0$

Ex: $y_{t+2} - y_{t+1} - y_t = 0 \iff y_{t+2} = y_{t+1} + y_t$

"each term is the sum of the previous two"

Char. eqn: $r^2 - r - 1 = 0$

$$r = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$r_1 = \frac{1+\sqrt{5}}{2} \qquad r_2 = \frac{1-\sqrt{5}}{2}$$

$\Downarrow$

$$\underline{\underline{y_t = c_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^t + c_2 \cdot \left(\frac{1-\sqrt{5}}{2}\right)^t}}$$

Why does this work? $y_{t+2} + a y_{t+1} + b y_t = 0$

Guess $y_t = r^t$

(differential eqn: e^{rt})

$$\left.\begin{aligned} y_t &= r^t \\ y_{t+1} &= r^{t+1} = r^t \cdot r \\ y_{t+2} &= r^{t+2} = r^t \cdot r^2 \end{aligned}\right\} \quad \begin{aligned} & r^t \cdot r^2 + a \cdot r^t \cdot r + b \cdot r^t = 0 \\ & r^t \cdot (r^2 + ar + b) = 0 \\ & \boxed{r^2 + ar + b = 0} \end{aligned}$$

Char. eqn.

$r_1 \neq r_2$ solutions of char eqn.

$\Rightarrow$ $\underline{y_t = c_1 \cdot r_1^t + c_2 \cdot r_2^t}$ is the general solution

$$y_{t+2} + a y_{t+1} + b y_t = 0$$

$$r^2 + ar + b = 0$$

$$r = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

i) Two roots $r_1 \neq r_2$: $\quad y_t = C_1 \cdot r_1^t + C_2 \cdot r_2^t$

ii) One double root $r_1 = r_2 = -a/2$: $\quad y_t = C_1 \cdot r_1^t + C_2 t r_1^t = (C_1 + C_2 t) r_1^t$

iii) No real roots:

$$y_t = (\sqrt{b})^t \cdot (C_1 \cdot \cos \theta t + C_2 \sin \theta t)$$

where $\theta = \cos^{-1}(-a/2\sqrt{b})$

Ex: $y_{t+2} - 4y_{t+1} + 4y_t = 0 \iff y_{t+2} = 4y_{t+1} - 4y_t$

$$r^2 - 4r + 4 = 0$$

$$(r-2)^2 = 0$$

$$r_1 = r_2 = 2 \longrightarrow y_t = C_1 \cdot 2^t + C_2 \cdot t \cdot 2^t = (C_1 + C_2 t) \cdot 2^t$$

Ex: $y_{t+2} - 12 y_{t+1} + 35 y_t = 0$

$$r^2 - 12r + 35 = 0$$

$$r = \frac{12 \pm \sqrt{144 - 4 \cdot 35}}{2}$$

$$= \frac{12 \pm 2}{2} = 5, 7$$

$$y_t = C_1 \cdot 5^t + C_2 \cdot 7^t$$

Ex:

$$y_{t+2} = y_{t+1} + y_t \quad , \; y_0 = y_1 = 1$$

$$y_{t+2} - y_{t+1} - y_t = 0$$

$$r^2 - r - 1 = 0$$

$$r = \frac{1 \pm \sqrt{5}}{2} \longrightarrow y_t = C_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^t + C_2 \cdot \left(\frac{1-\sqrt{5}}{2}\right)^t$$

$y_0 = 1$:

$$1 = C_1 \cdot (\ldots)^0 + C_2 \cdot (\ldots)^0 .$$

$$1 = C_1 + C_2 \Longrightarrow C_2 = 1 - C_1$$

$y_1 = 1$:

$$1 = C_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^1 + C_2 \left(\frac{1-\sqrt{5}}{2}\right)^1 \quad | \cdot 2$$

$$2 = C_1 \cdot (1+\sqrt{5}) + (1-C_1) \cdot (1-\sqrt{5})$$

$$2 = C_1 \cdot (\cancel{1} + \sqrt{5} - \cancel{1} + \sqrt{5}) \cdot + (1-\sqrt{5})$$

$$2 - (1-\sqrt{5}) = C_1 \cdot 2\sqrt{5}$$

$$C_1 = \frac{1+\sqrt{5}}{2\sqrt{5}} \qquad C_2 = \frac{2\sqrt{5} - (1+\sqrt{5})}{2\sqrt{5}}$$

$$C_2 = \frac{\sqrt{5} - 1}{2\sqrt{5}}$$

$$y_t = \frac{1+\sqrt{5}}{2\sqrt{5}} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^t + \frac{\sqrt{5}-1}{2\sqrt{5}} \cdot \left(\frac{1-\sqrt{5}}{2}\right)^t$$

Inhomogeneous case: $y_{t+2} + a y_{t+1} + b y_t = f_t$

General solution: $y_t = y_t^h + y_t^p$

Ex: $y_{t+2} - y_{t+1} - 2y_t = 3^t$

$$y_t = y_t^h + y_t^p = c_1 \cdot 2^t + c_2 \cdot (-1)^t + \frac{1}{4} \cdot 3^t$$

y_t^h: $y_{t+2} - y_{t+1} - 2y_t = 0$

$$r^2 - r - 2 = 0$$

$$r = \frac{+1 \pm \sqrt{1+8}}{2}$$

$$= \frac{1 \pm 3}{2} = 2, -1 \longrightarrow y_t^h = c_1 \cdot 2^t + c_2 \cdot (-1)^t$$

y_t^p: $y_{t+2} - y_{t+1} - 2y_t = 3^t$

$f_t = 3^t$

$f_{t+1} = 3^{t+1} = 3 \cdot 3^t$

$f_{t+2} = 3^{t+2} = 9 \cdot 3^t$

Guess: $y_t = A \cdot 3^t$

$y_{t+1} = 3A \cdot 3^t$

$y_{t+2} = 9A \cdot 3^t$

$:3^t \,|\quad 9A \cdot 3^t - 3A \cdot 3^t - 2 \cdot A \cdot 3^t = 3^t$

$$9A - 3A - 2A = 1$$

$$4A = 1$$

$$A = 1/4 \longrightarrow y_t^p = \frac{1}{4} \cdot 3^t$$

Method: * Choose a guess y_t such that it depends on parameters and such that it has a form that generalizes f_t, f_{t+1}, f_{t+2}

* put the guess into the equation
* if the initial guess doesn't work, try to multiply it with t.

Ex: $y_{t+2} - 2y_{t+1} + y_t = -15$

$$y_t = y_t^h + y_t^P = C_1 \cdot 1^t + C_2 t \cdot 1^t + y_t^P$$
$$= C_1 + C_2 t + y_t^P$$

y_t^P: $y_{t+2} - 2y_{t+1} + y_t = -15$

$f_t = -15$

$f_{t+1} = -15$

$f_{t+2} = -15$

Guess: $y_t = At^2$

$y_{t+1} = A(t+1)^2$
$= A(t^2+2t+1)$

$y_{t+2} = A(t+2)^2$
$= A(t^2+4t+4)$

$$A(t^2+4t+4) - 2 \cdot A(t^2+2t+1) + 1 \cdot At^2 = -15$$

$$0 \cdot t^2 + 0 \cdot t + (2A) = -15$$

$$A = \frac{-15}{2} = -7.5$$

$$y_t^P = -7.5t^2$$

② Stability:

Differential equation $\longrightarrow$ $y = y(t)$ solution

Difference equation $\longrightarrow$ $y = y_t$ solution

It is called globally asymptotically stable

if

$$\bar{y} = \lim_{t\to\infty} y = \begin{cases} \lim\limits_{t\to\infty} y(t) \\ \lim\limits_{t\to\infty} y_t \end{cases}$$

satisfy i) $\bar{y}$ is finite ii) $\bar{y}$ does not depend on the undetermined coefficients

Interpretation: There is a long-run equilibrium $\bar{y}$, which is finite and independent of initial conditions.

Ex: $y_{t+2} - y_{t+1} - 2y_t = 3^t$

$$y_t = C_1 \underbrace{2^t}_{\to\infty} + C_2 \underbrace{(-1)^t}_{\pm 1} + \frac{1}{4} 3^t \longrightarrow \infty$$

(unstable) (as $t \to \infty$)

not stable

Ex: $y'' - 3y' + 2y = 4$

$$y = y_h + y_p = C_1 \cdot e^{t} + C_2 \cdot e^{2t} + 2 \to \pm\infty$$

(unless $C_1 = C_2 = 0$)

y_h: $r^2 - 3r + 2 = 0$

$r = 1,\ r = 2$

y_p: $y = A = 2$

not globally asymptotically stable

For globally asymptotical stability, it is necessary that "essentially"

$$\boxed{\begin{array}{lll} |r| < 1 & \text{for} & \text{difference eqn's} \\ r < 0 & \text{"} & \text{differential equations} \end{array}}$$

$r^t \to 0$

$e^{rt} \to 0$

Ex: Systems of difference equations

$$\boxed{\begin{aligned} x_{t+1} &= 2x_t - y_t \\ y_{t+1} &= x_t \end{aligned}}$$

similar examples are Markov chains.

$$x_{t+1} = 2x_t - x_{t-1}$$

$$x_{t+1} - 2x_t + x_{t-1} = 0$$

$$r^2 - 2r + 1 = 0$$

$$r = 1 \longrightarrow x_t = (c_1 + c_2 t) \cdot 1^t = C_1 + C_2 t$$

$$y_t = x_{t-1} = c_1 + c_2(t-1)$$

$$= C_1 + C_2 t - C_2$$