LECTURE 11 (F) GRA 6035 MATHEMATICS

EIVIND ERIKSEN NOV 03, 2014

Plan:

1. Exact differential equations
2. Second order linear diff. eqn's
   a) Homogeneous
   b) Inhomogeneous

Reading: [ME] 24.1–24.3, (24.4–24.6)

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1 Exact differential equations

Ex: $3t^2 + y^2 + (2ty-2)y' = 0$, $y(1) = 2$

$(2ty-2)y' = -(3t^2+y^2)$

$y' = -\dfrac{3t^2+y^2}{2ty-2}$

Linear? No

$b(t) - a(t)\cdot y$

If it was lin.:

$y' + a(t)y = b(t)$

Use int. factor:

$e^{\int a(t)dt}$

Exact?

$\underbrace{(3t^2+y^2)}_{=p} + \underbrace{(2ty-2)}_{=q}y' = 0$

Find $h = h(y,t)$ such that

$$\boxed{\begin{aligned} h'_t &= p = 3t^2+y^2 \\ h'_y &= q = 2ty-2 \end{aligned}}$$

Separable? No.

$f(t)\cdot g(y)$

$\frac{1}{g(y)}\cdot y' = f(t)$

In that case, general solution is

$h(y,t) = C$

① $h'_t = 3t^2 + y^2$
② $h'_y = 2ty - 2$

① $h'_t = 3t^2 + y^2 \Rightarrow h = \int 3t^2 + y^2 \, dt$

$$= \underline{t^3 + y^2 t + C(y)}$$

② $h'_y = \underline{0 + 2yt} + C'(y) = \underline{2ty} - 2$

$$C'(y) = -2$$

$$C(y) = -2y$$

$h = t^3 + y^2 t - 2y$ satisfies (1) and (2)

$\shortparallel$

General solution: (implicit form) $\boxed{t^3 + y^2 t - 2y = C}$

Initial condition: $y(1) = 2 \Leftrightarrow$ $\begin{matrix} t=1 \\ y=2 \end{matrix}$ satisfies eqn.

$$1^3 + 2^2 \cdot 1 - 2 \cdot 2 = C$$

$$\underline{C = 1}$$

$$t^3 + y^2 t - 2y = 1$$

$$t \cdot y^2 - 2 \cdot y + (t^3 - 1) = 0$$

$$y = \frac{2 \pm \sqrt{4 - 4 \cdot t \cdot (t^3 - 1)}}{2t} = \frac{2 \pm \sqrt{4 + 4t - 4t^4}}{2t}$$

$$= \frac{2 + \sqrt{4 + 4t - 4t^4}}{2t}$$

② Second order differential equations

<u>Ex:</u>

$$y'' = 6t - 2$$

$$y' = \int 6t - 2 \, dt = 3t^2 - 2t + C_1$$

$$y = \int 3t^2 - 2t + C_1 \, dt$$

$$y = \underline{\underline{t^3 - t^2 + C_1 t + C_2}} \quad \text{general Solution}$$

<u>Second order diff. equ:</u>

Contains y'', typically contains y'', y', y, t

there will be two undetermined constants in the general solution.

<u>Ex:</u> $y'' = 6t - 2$, $y(0) = 1$, $y'(0) = 2$

$$y = t^3 - t^2 + C_1 t + C_2 \qquad y' = 3t^2 - 2t + C_1$$

$\underline{y(0) = 1}$: $\quad 1 = C_2 \quad \Rightarrow C_2 = \underline{1}$

$\underline{y'(0) = 2}$: $\quad 2 = C_1 \quad \Rightarrow C_1 = \underline{2}$

$$y = \underline{\underline{t^3 - t^2 + 2t + 1}}$$

Linear second order diff. eqn. with constant coeffs:

Defn: $$\boxed{y'' + ay' + by = f(t)}$$

a, b: constants

$f(t)$: expression in t

a) Homogeneous case: $y'' + ay' + by = 0$

Solution method:

Ex: $y'' - 3y' + 2y = 0 \rightsquigarrow$ Char. eqn:

$r^2 - 3r + 2 = 0$

$r = 1, \quad r = 2$

e^{rt} for $r = 1, 2$

$y = C_1 \cdot e^t + C_2 \cdot e^{2t} \leftarrow$

Why does this work?

$y'' + ay' + by = 0 \leftarrow$ Try $y = e^{rt}$

$y' = e^{rt} \cdot r$

$y'' = e^{rt} \cdot r^2$

$r^2 e^{rt} + a \cdot r e^{rt} + b e^{rt} = 0 \quad | : e^{rt}$

$$\boxed{r^2 + ar + b = 0}$$

char. eqn.

Homogeneous case: $y'' + ay' + by = 0$

Consider the characteristic eqn. $r^2 + ar + b = 0$ and find its solutions:

i) Two roots $r_1 \neq r_2$: ($a^2 - 4b > 0$) $\quad y = C_1 \cdot e^{r_1 t} + C_2 \cdot e^{r_2 t}$

ii) one double root r: ($a^2 - 4b = 0$) $\quad y = C_1 e^{rt} + C_2 t \cdot e^{rt}$

iii) No roots: ($a^2 - 4b < 0$) $\quad y = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t)$

where $\alpha = -\frac{a}{2}$ $\beta = \frac{\sqrt{4b - a^2}}{2}$

Ex: $y'' - 4y' + 4y = 0$

Char. eqn: $r^2 - 4r + 4 = 0$

$(r-2)^2 = 0$

$r = 2$ (double root) $\Rightarrow$

General solution:

$y = C_1 e^{2t} + C_2 t e^{2t}$

$= (C_1 + C_2 t) e^{2t}$

Ex: $y'' + y = 0$

Char. eqn: $r^2 + 1 = 0$

$r = \pm\sqrt{-1}$

no real roots

$\alpha = 0, \beta = 1 \Rightarrow y = C_1 \cdot \cos t + C_2 \cdot \sin t$

b) **Nonhomogeneous case** : $y'' + ay' + by = f(t)$

Ex: $y'' - 3y' + 2y = 4$

Superposition principle: $\boxed{y'' + ay' + by = f(t)}$

The general solution $y = y_h + y_p$

where y_h is the general solution of the homogeneous diff. eqn. $y'' + ay' + by = 0$ and y_p is a particular solution of $y'' + ay' + by = f(t)$.

$y'' - 3y' + 2y = 4$:

$$y = y_h + y_p = \underline{\underline{c_1 e^t + c_2 e^{2t} + 2}}$$

y_h: $y'' - 3y' + 2y = \underline{0}$

$$r^2 - 3r + 2 = 0$$

$$\underline{r = 1}, \ \underline{r = 2} \longrightarrow y_h = c_1 e^t + c_2 e^{2t}$$

y_p: $y'' - 3y' + 2y = 4$

$$\left.\begin{array}{l} y = C \\ y' = 0 \\ y'' = 0 \end{array}\right\} \quad \begin{array}{r} 0 - 3 \cdot 0 + 2 \cdot C = 4 \\ C = 2 \\ \underline{\underline{y_p = 2}} \end{array}$$

How to find y_p?

Ex: $y'' - 4y' + 3y = e^{2t}$

$$y = y_h + y_p = \underline{\underline{c_1 e^{3t} + c_2 e^{t} - e^{2t}}}$$

y_h: $y'' - 4y' + 3y = 0$

$$r^2 - 4r + 3 = 0$$

$$r = \frac{4 \pm \sqrt{16 - 4 \cdot 3}}{2}$$

$$r = \frac{4 \pm 2}{2}$$

$$r_1 = \underline{3}, \quad r_2 = \underline{1} \implies y_h = \underline{c_1 \cdot e^{3t} + c_2 e^{t}}$$

y_p: $y'' - 4y' + 3y = e^{2t}$

$$\text{Look at} \quad \left.\begin{aligned} f(t) &= e^{2t} \\ f'(t) &= 2 \cdot e^{2t} \\ f''(t) &= 4 \cdot e^{2t} \end{aligned}\right\} \quad y_p = C \cdot e^{2t}$$

choose a guess (with a parameter) that "looks like" f, f', f''.

$$\left.\begin{aligned} y &= c e^{2t} \\ y' &= 2c e^{2t} \\ y'' &= 4c e^{2t} \end{aligned}\right\} \quad \begin{aligned} &4c \cdot e^{2t} - 4 \cdot (2c e^{2t}) + 3(c e^{2t}) = e^{2t} \\ &(4c - 8c + 3c) e^{2t} = e^{2t} \quad | : e^{2t} \end{aligned}$$

$$-c = 1$$

$$c = -1$$

$$y_p = c e^{2t} = \underline{-e^{2t}}$$

Ex: $y'' - y' = t$

General solution: $y = y_h + y_p = \underline{\underline{c_1 + c_2 e^t + \frac{1}{2}t^2 - t}}$

y_h: $y'' - y' = 0$

$r^2 - r = 0$

$r = 0, r = 1 \rightsquigarrow y_h = c_1 \cdot e^{0t} + c_2 e^{1 \cdot t}$

$= \underline{c_1 + c_2 e^t}$

y_p: $y'' - y' = t$

$\left.\begin{array}{l} f = t \\ f' = 1 \\ f'' = 0 \end{array}\right\}$ $y_p = At + B$ (A, B const.)

$\left.\begin{array}{l} y = At + B \\ y' = A \\ y'' = 0 \end{array}\right\}$ $0 - A = t$

no solution

If the initial guess does not work, try to multiply it by t.

$y = (At + B) \cdot t = \underline{At^2 + Bt}$

$y' = 2At + B$

$y'' = 2A$

$2A - (2At + B) = t$

$(2A - B) + (-2A)t = t$

$2A - B = 0 \quad B = -1$

$-2A = 1 \quad \Rightarrow A = -1/2$

$y_p = \underline{-\frac{1}{2}t^2 - t}$

Superposition principle:

If $y_1(t)$ is solution of $y'' + ay' + by = f_1(t)$

$y_2(t)$ — " — $y'' + ay' + by = f_2(t)$

then $y_1(t) + y_2(t)$ is a solution of $y'' + ay' + by = f_1(t) + f_2(t)$

Check: Put $y_1 + y_2$ into the last eqn.:

$$(y_1'' + y_2'') + a(y_1' + y_2') + b \cdot (y_1 + y_2)$$

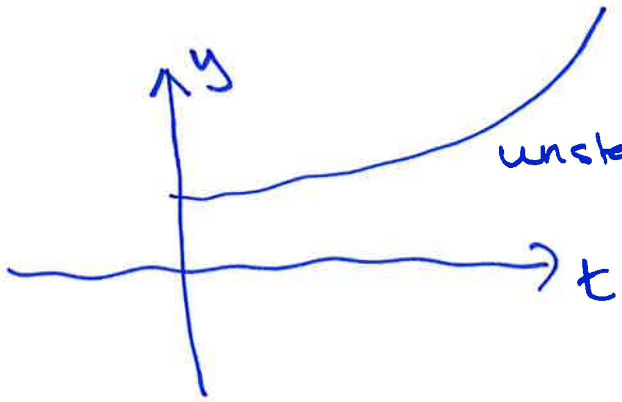
$$= (y_1'' + ay_1' + by_1) + (y_2'' + ay_2' + by_2)$$

$$= f_1(t) + f_2(t) \qquad \square$$

Stability:

Ex: $y'' - 4y' + 3y = e^{2t}$

$y = y_h + y_p = \underline{c_1 e^{3t} + c_2 e^{t} - e^{2t}}$



as $t \to \infty$,

$e^{3t} \to \infty$

$e^{t} \to \infty$

If $y = y_h + y_p = c_1 e^{r_1 t} + c_2 e^{r_2 t} + y_p$

and $\underbrace{r_1, r_2 < 0}_{\text{stable}}$, then $\lim_{t \to \infty} y(t) = \lim_{t \to \infty} y_p$

Ex: $y'' - 7y' + 12y = te^t$

$$y = y_h + y_p = \underbrace{c_1 e^{3t} + c_2 e^{4t}}_{y_h} + y_p$$

$y_p = ?$

$y = \underline{(At+B)e^t}$

$y' = A\cdot e^t + (At+B)e^t$

$= \underline{(At+B+A)e^t}$

$y'' = A\cdot e^t + (At+B+A)e^t$

$= \underline{(At+B+2A)e^t}$

$\underline{f = te^t}$

$f' = 1\cdot e^t + t\cdot e^t$

$= (t+1)e^t$

$f'' = 1\cdot e^t + (t+1)e^t$

$= (t+2)e^t$

$$(At+B+2A)e^t - 7\cdot(At+B+A)e^t + 12(At+B)e^t = te^t$$

$$(At+B+2A) - 7(At+B+A) + 12(At+B) = t$$

$$\underbrace{(6A)}_{=1}t + \underbrace{(B+2A-7B-7A+12B)}_{=0} = t$$

$\underline{A = 1/6}$

$6B - 5A = 0$

$B = \frac{5A}{6} = \frac{5}{6}\cdot\frac{1}{6} = \underline{\frac{5}{36}}$

$$y_p = (At+B)e^t = \underline{\underline{\left(\frac{1}{6}t + \frac{5}{36}\right)e^t}}$$

Ex: $y' - 2y = t^2$

General solution: $y = y_h + y_p$

y_h: $y' - 2y = 0$

$r - 2 = 0$

$r = 2 \longrightarrow y_h = C \cdot e^{2t}$

y_p: $y' - 2y = t^2$

$f = t^2$

$f' = 2t$

$y = At^2 + Bt + C$

$y' = A \cdot 2t + B$

$(2At + B) - 2(At^2 + Bt + C) = t^2$

$\underbrace{(-2A)}_{=1}t^2 + \underbrace{(2A - 2B)}_{=0}t + \underbrace{(B - 2C)}_{=0} = t^2$

$A = -1/2$ $\quad B = -1/2$ $\quad C = B/2 = -1/4$

$$y = C \cdot e^{2t} + \left(-\tfrac{1}{2}t^2 - \tfrac{1}{2}t - 1/4\right)$$

could also use int. factor.