LECTURE 11 (B)

GRA 6035 MATHEMATICS

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Plan:

1. Exact differential equations
2. Second order linear diff. eqn's
   a) Homogeneous b) Inhomogeneous

Reading:

[ME] 24.1–24.3, (24.4–24.6)

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1 Exact differential equations

Ex:

$$3t^2+y^2+(2ty-2)y'=0, \quad y(1)=2$$

$$y' = \frac{-(3t^2+y^2)}{2ty-2} = F(t,y)$$

a) Is $F = b(t) - a(t)\cdot y$? **Linear**

Not linear in this case.

$y' + a(t)y = b(t) \longrightarrow$ use integrating factor $e^{\int a(t)\,dt}$

b) Is $F = f(y)\cdot g(t)$? **Seperable**

Not seperable in this case.

$y' = f(y)\cdot g(t) \longrightarrow \int \frac{1}{f(y)} y'\,dt = \int g(t)\,dt$

$$\underbrace{3t^2+y^2}_{p} + \underbrace{(2ty-2)}_{q}\, y' = 0$$

Try to find $h = h(y,t)$ such that $\begin{cases} h'_t = p \\ h'_y = q \end{cases}$

$$\boxed{\begin{aligned} &(1)\quad h'_t = 3t^2+y^2 \\ &(2)\quad h'_y = 2ty-2 \end{aligned}}$$

① $h'_t = 3t^2+y^2$

$$h = \int 3t^2+y^2\, dt$$

$$= \underline{t^3 + y^2 t + c(y)}$$

② $h'_y = 2ty-2$

$$\left(t^3+y^2t+c(y)\right)'_y = 2ty-2$$

$$0 + \underline{2yt} + c'(y) = \underline{2ty} - 2$$

$$c'(y) = -2$$

$$c(y) = -2y$$

Solution of (1) and (2): $h = \underline{t^3+y^2t-2y}$

$\Rightarrow$ Solution of the diff. eqn is

$$h = C$$

$$t^3+y^2t-2y = C$$

$$t^3+y^2t-2y = 1$$

implicit form

Initial condition

$y(1) = 2$

$t=1 \quad y=2$

$1^3 + 2^2\cdot 1 - 2\cdot 2 = C$

$\underline{C=1}$

$$t^3 + y^2 t - 2y = 1$$

$$\underset{\overset{\shortparallel}{a}}{t} \cdot y^2 - \underbrace{2}_{\overset{\shortparallel}{b}} \cdot y + \underset{\overset{\shortparallel}{c}}{(t^3 - 1)} = 0$$

$$y = \frac{2 \pm \sqrt{4 - 4 \cdot t \cdot (t^3 - 1)}}{2t}$$

$$= \frac{2 \pm \sqrt{4 + 4t - 4t^4}}{2t}$$

$y(1) = 2$

$$= \frac{2 + \sqrt{4 + 4t - 4t^4}}{2t}$$

② Second order differential equations

Ex:

$$y'' = 6t-2$$

$$y' = \int 6t-2 \, dt = 3t^2 - 2t + C_1$$

$$y = \int 3t^2 - 2t + C_1 \, dt$$

$$y = \underline{t^3 - t^2 + C_1 t + C_2}$$

Second order differential equations:

* contains y'', typically contains y'', y', y, t
* General form: $y'' = F(y', y, t)$
* there will be <u>two</u> undetermined constants

$\underline{y'' = 6t-2}$, $y(0)=1$, $y'(0)=2$

$$y = t^3 - t^2 + C_1 t + C_2 = \underline{\underline{t^3 - t^2 + 2t + 1}}$$

$\underline{y(0)=1}$: $1 = 0^3 - 0^2 + C_1 \cdot 0 + C_2$

$\underline{C_2 = 1}$

$\underline{y'(0)=2}$: $y' = 3t^2 - 2t + C_1$

$2 = 3 \cdot 0^2 - 2 \cdot 0 + C_1$

$\underline{C_1 = 2}$

Linear second order diff. eqn. with Constant coefficients:

$$y'' + a\cdot y' + by = f(t)$$

where a, b constants and f(t) is a function of t.

i) Homogeneous case: $f(t) = 0$

ii) Inhomogeneous case: $f(t) \neq 0$

i) Homogeneous case

$$y'' + ay' + by = 0 \qquad (a, b \text{ constants})$$

Ex: $y'' - 3y' + 2y = 0$

Characteristic equation: $r^2 - 3r + 2 = 0$

$r = 1$, $r = 2$

$\Downarrow$

General solution of diff. eqn:

$$y = C_1 \cdot e^{1\cdot t} + C_2 \cdot e^{2t}$$

$$y = C_1 e^{t} + C_2 e^{2t}$$

Why does this work?

$$y'' + ay' + by = 0 \longleftarrow \text{Try} \begin{cases} y = e^{rt} \\ y' = e^{rt} \cdot r \\ y'' = e^{rt} \cdot r^2 \end{cases}$$

$$r^2 \cdot e^{rt} + a \cdot (re^{rt}) + be^{rt} = 0$$

$$e^{rt} \cdot (r^2 + ar + b) = 0$$

Char. eqn. $\boxed{r^2 + ar + b = 0}$

r root in the characteristic equation $\Longleftrightarrow$ $y = e^{rt}$ is a solution of the diff. eqn.

Superposition principle for linear diff. eqn:

If y_1 is a solution of $y'' + ay' + by = f_1(t)$

and y_2 — " — $y'' + ay' + by = f_2(t)$

then $c_1y_1 + c_2y_2$ — " — $y'' + ay' + by = c_1f_1(t) + c_2f_2(t)$

In particular, if y_1 and y_2 are solutions of $y'' + ay' + by = 0$, then $c_1y_1 + c_2y_2$ is a solution also.

General method in the homogeneous case

$$y'' + ay' + by = 0$$

Char. eqn: $r^2 + ar + b = 0$

$$r = \frac{-a \pm \sqrt{a^2-4b}}{2}$$

Cases:

a) $a^2-4b>0$: Two root $r_1 \neq r_2$

General sol. of diff.eqn: $y = \underline{C_1 e^{r_1 t} + C_2 e^{r_2 t}}$

b) $a^2-4b=0$: One double root $r_1 = r_2$ $(= -a/2)$

General sol. of diff.eqn: $y = \underline{C_1 e^{r_1 t} + C_2 t e^{r_2 t}}$

c) $a^2-4b<0$: No roots

General sol: $y = e^{\alpha t} \cdot (C_1 \cdot \cos \beta t + C_2 \cdot \sin \beta t)$

$$\alpha = -\frac{a}{2} \qquad \beta = \frac{\sqrt{4b-a^2}}{2}$$

Ex: $y'' - 6y' + 9y = 0$

$$\left.\begin{array}{l} r^2 - 6r + 9 = 0 \\ r = \dfrac{6 \pm \sqrt{36-4\cdot 9}}{2} \\ r = 3 \text{ (double root)} \end{array}\right\} \quad \begin{aligned} y &= C_1 \cdot e^{3t} + C_2 \cdot t e^{3t} \\ &= \underline{\underline{(C_1 + C_2 t) e^{3t}}} \end{aligned}$$

Ex: $y'' + y = 0$

$$r^2 + 1 = 0$$
$$r = \pm\sqrt{-1}$$
no roots

$$y = e^{0\cdot t}\cdot(C_1\cdot \cos t + C_2 \sin t)$$
$$= C_1 \cdot \cos \cancel{t} + C_2 \cdot \sin t$$

$$\alpha = -\frac{a}{2} = 0$$
$$\beta = \frac{\sqrt{4b-a^2}}{2} = 1$$

ii) Inhomogeneous case: $y'' + ay' + by = f(t)$

Superposition principle:

If y_1 is a solution of $y'' + ay' + by = 0$

and y_2 — " — $y'' + ay' + by = f(t)$

then $y_1 + y_2$ — " — $y'' + ay' + by = f(t)$

The general solution of $y'' + ay' + by = f(t)$ is given by $y = y_h + y_p$ where

y_h : general solution of $y'' + ay' + by = 0$

y_p : a particular solution of $y'' + ay' + by = f(t)$

Ex:

$$y'' - 4y' + 3y = 12$$

$$y = y_h + y_p = \underbrace{c_1 e^{3t} + c_2 e^{t}}_{y_h} + \underbrace{4}_{y_p}$$

y_h:

$$y'' - 4y' + 3y = 0$$

$$r^2 - 4r + 3 = 0$$

$$r = 3 \, , \; r = 1 \quad \rightarrow \quad y_h = c_1 \cdot e^{3t} + c_2 e^{t}$$

y_p:

$$y'' - 4y' + 3y = 12$$

$$\left.\begin{aligned} y &= C \\ y' &= 0 \\ y'' &= 0 \end{aligned}\right\} \quad 0 - 4 \cdot 0 + 3 \cdot C = 12$$

$$C = 4$$

$$y_p = 4$$

Ex: $y'' - 4y' + 3y = e^{2t}$

$$y = y_h + y_p = \underbrace{c_1 e^{3t} + c_2 e^{t}}\ \dot{+}\ \underline{e^{2t}}$$

y_h: $y'' - 4y' + 3y = 0$

$r^2 - 4r + 3 = 0$

$r = 3,\ r = 1$ $\qquad y_h = \underline{c_1 e^{3t} + c_2 e^{t}}$

y_p: How do we guess y_p?

$y'' - 4y' + 3y = e^{2t}$

Look at $f(t) = e^{2t}$ $\Rightarrow$ Guess:

$f' = 2e^{2t}$ $\qquad y = c \cdot e^{2t}$

$f'' = 4e^{2t}$ $\qquad y' = 2c\, e^{2t}$

$y'' = 4c\, e^{2t}$

$y'' - 4y' + 3y = e^{2t}$

$(4ce^{2t}) - 4 \cdot (2ce^{2t}) + 3(ce^{2t}) = e^{2t}$

$(4c - 8c + 3c) \cdot e^{2t} = e^{2t}$

$-c \cdot e^{2t} = e^{2t}$

$c = -1$

$y_p = c \cdot e^{2t} = \underline{-e^{2t}}$

Ex: $y'' - y' = t$

$$y = y_h + y_p = \underline{c_1 + c_2 e^t - \tfrac{1}{2}t^2 - t}$$

y_h: $r^2 - r = 0$

$r = 0, r = 1 \quad \rightarrow \quad y_h = c_1 e^{0t} + c_2 e^t$

$= \underline{c_1 + c_2 e^t}$

y_p: $f = t$, $f' = 1$, $f'' = 0$

$$\boxed{\begin{aligned} y &= \underline{At + B} \\ y' &= A \\ y'' &= 0 \end{aligned}}$$

$0 - A = t$

no solutions

If the initial guess does not work, try to multiply it with t

$$\left.\begin{aligned} y &= \underline{At^2 + Bt} \\ y' &= 2At + B \\ y'' &= 2A \end{aligned}\right\} \quad \begin{aligned} &(2A) - (2At + B) = t \\ &\underbrace{(-2A)}_{1} \cdot t + \underbrace{(2A - B)}_{0} = t \end{aligned}$$

$$y_p = \underline{-\tfrac{1}{2}t^2 - t} \Leftarrow \begin{cases} -2A = 1 \Rightarrow A = -1/2 \\ 2A - B = 0 \Rightarrow B = 2A = -1 \end{cases}$$

Ex: $y' - 2y = t^2$ ← First order linear const. coeff.

$$y = y_h + y_p = c_1 \cdot e^{2t} + -\frac{1}{2}t^2 - \frac{1}{2}t - \frac{1}{4}$$

y_h: $y' - 2y = 0$

$r - 2 = 0$

$r = 2 \rightarrow y_h = c_1 \cdot e^{2t}$

y_p: $f = t^2$ $\quad y = At^2 + Bt + C$

$f' = 2t$ $\quad y' = 2At + B$

$f'' = 2$

Alt: $y' - 2y = t^2$

$(y \cdot e^{-2t})' = t^2 e^{-2t}$

$y e^{-2t} = \int t^2 e^{-2t} dt$

$(2At + B) - 2 \cdot (At^2 + Bt + C) = t^2$

$\underbrace{(-2A)}_{=1} \cdot t^2 + \underbrace{(2A - 2B)}_{=0} t + \underbrace{(B - 2C)}_{=0} = t^2$

$A = -1/2 \quad B = -1/2 \quad C = -1/4$

$y_p = -\frac{1}{2}t^2 - \frac{1}{2}t - \frac{1}{4}$

Ex: $y'' - 7y' + 12y = te^t$

$$y = y_h + y_p = \underbrace{c_1 e^{3t} + c_2 e^{4t}}_{y_h} + y_p$$

$y_p = ?$

$$y = \underline{(At+B)e^t}$$

$$y' = A \cdot e^t + (At+B)e^t$$

$$= \underline{(At+B+A)e^t}$$

$$y'' = A \cdot e^t + (At+B+A)e^t$$

$$= \underline{(At+B+2A)e^t}$$

$$\underline{f = te^t}$$

$$f' = 1 \cdot e^t + t \cdot e^t$$

$$= (t+1)e^t$$

$$f'' = 1 \cdot e^t + (t+1)e^t$$

$$= (t+2)e^t$$

$$(At+B+2A)e^t - 7 \cdot (At+B+A)e^t + 12(At+B)e^t = te^t$$

$$(At+B+2A) - 7(At+B+A) + 12(At+B) = t$$

$$\underbrace{(6A)}_{=1}t + (\underbrace{B+2A-7B-7A+12B}_{=0}) = t$$

$$A = \underline{1/6} \qquad 6B - 5A = 0$$

$$B = \frac{5A}{6} = \frac{5}{6} \cdot \frac{1}{6} = \underline{\frac{5}{36}}$$

$$y_p = (At+B)e^t = \underline{\underline{\left(\frac{1}{6}t + \frac{5}{36}\right)e^t}}$$