What is a differential equation?

- Separable differential equations
- First order linear diff. eq.
- Exact diff. eq.

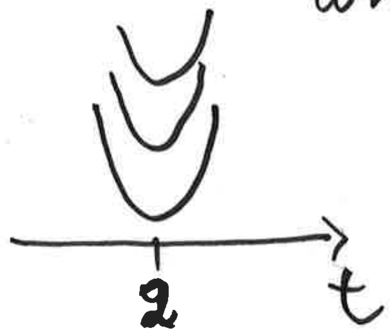
A diff. equation is an equation where the unknown is a function. The derivative of the function is included.

Ex. Find the function $y(t)$ such that

$$y'(t) = 2t - 4$$

$$y(t) = \int 2t - 4 \, dt = \underline{t^2 - 4t + C}$$

where C is any constant.



$y(t) = t^2 - 4t + C$ is the general solution

Suppose we have the initial value $y(0) = 4$ this gives the particular solution

$4 = 0^2 - 4 \cdot 0 + C \iff C = 4$

part. sol.:

$$y(t) = \underline{\underline{t^2 - 4t + 4}}$$

ex. Find $y(t)$ when

$$y'(t) = e^{2t} \qquad y(0) = 5$$

General solution

$$\underline{\underline{y(t) = \frac{1}{2}e^{2t} + c}}$$

$y(0) = 5$ gives

$$5 = \frac{1}{2} \cdot 1 + c$$

$$c = \frac{9}{2}$$

Particular solution

$$\underline{\underline{y(t) = \frac{1}{2}e^{2t} + \frac{9}{2}}}$$

Separable differential equations

A differential first order equation is called <u>separable</u> if it can be written $\boxed{y' = f(y)\cdot g(t)}$

examples:

i) $y' = 2t - 4$ $\quad y' = \underset{\uparrow\, f(y)}{1} \cdot \underbrace{(2t-4)}_{g(t)}$ separable

ii) $y' = \cancel{5t}\, 5y$ $\quad y' = \underset{\uparrow\, f(y)}{5y} \cdot \underset{\uparrow\, g(t)}{1}$ — " —

iii) $y' = y + t$ not separable

iv) $t \cdot y' = y^2 \cdot (t-1)$

$y' = y^2 \cdot \frac{t-1}{t}$ separable.

How do we solve separable diff. eq.s ?

ex: $y' = 5y$

$\frac{1}{y}\cdot y' = 5$ $\qquad y' = \frac{dy}{dt}$

$\frac{1}{y}\cdot \frac{dy}{dt} = 5$

$\frac{1}{y}\, dy = 5\, dt$

$$\int \frac{1}{y}\,dy = \int 5\,dt$$

$$\ln|y| + C_1 = 5t + C_2$$

$$\ln|y| = 5t + C \qquad (C = C_2 - C_1)$$

$$|y| = e^{5t + C} = e^{C} \cdot e^{5t}$$

$$y = \pm e^{C} e^{5t}$$

$$\underline{\underline{y = Ke^{5t}}}$$

e.

$$y' = t^2 \cdot y + y$$

$$y' = y(t^2 + 1) \qquad \text{(separable)}$$

$$\frac{1}{y} \cdot y' = t^2 + 1$$

$$\int \frac{1}{y}\,dy = \int (t^2 + 1)\,dt$$

$$\ln|y| = \frac{1}{3}t^3 + t + C$$

$$|y| = e^{\frac{1}{3}t^3 + t} \cdot e^{C}$$

$$\underline{\underline{y = K \cdot e^{\frac{1}{3}t^3 + t}}} \qquad \text{gen. sol.}$$

Control: $y' = K \cdot e^{\frac{1}{3}t^3 + t} \cdot (t^2 + 1)$ left.

$y(t^2 + 1) = Ke^{\frac{1}{3}t^3 + t}(t^2 + 1)$ right

Ex

$$y' = 3y^2 - 2ty^2$$

$$y' = y^2 \cdot (3-2t) \qquad | : y^2$$

$$\frac{1}{y^2}\frac{dy}{dt} = 3-2t$$

$$\int \frac{1}{y^2}\,dy = \int (3-2t)\,dt$$

$$-\frac{1}{y} = 3t - t^2 + C \qquad \text{implisit solution}$$

$$\Updownarrow$$

$$\frac{1}{y} = t^2 - 3t - C$$

$$\underline{\underline{y = \frac{1}{t^2-3t-C}}} \qquad \text{explisit solution}$$

General solution:

$$y' = f(y)\cdot g(t)$$

$$\frac{1}{f(y)}\frac{dy}{dt} = g(t)$$

$$\int \frac{1}{f(y)}\,dy = \int g(t)\,dt$$

Solving the integrals on both sides gives the implisit solution.

Solving with respect to y gives the explisit solution.

Linear 1. order differential equations

$$\boxed{y' + a(t)\cdot y = b(t)} \qquad y' = b(t) - a(t)\cdot y$$

If $a(t) = a$ and $b(t) = b$ (a and b are constants)
then the equation is called autonomous.

$$y' + ay = b$$

Ex

$$y' = 2 - 3y$$

$$y' + 3y = 2$$

Multiply by e^{3t} (integrating factor)
on both sides.

$$y'\cdot e^{3t} + 3y\cdot e^{3t} = 2\cdot e^{3t}$$

$$\left(y\cdot e^{3t}\right)' = 2\cdot e^{3t}$$

Integrate both sides with respect to t

$$y\cdot e^{3t} = \int 2e^{3t}\,dt$$

$$y\cdot e^{3t} = \frac{2}{3}e^{3t} + c \qquad \Big|\cdot e^{-3t}$$

$$\underline{\underline{y = \frac{2}{3} + ce^{-3t}}} \qquad \text{general solution.}$$

Ex: $\quad y' - 4y = 8t \qquad \Big| \cdot e^{-4t}$

$$y' \cdot e^{-4t} - 4y \cdot e^{-4t} = 8t \cdot e^{-4t}$$

$$\left(y \cdot e^{-4t}\right)' = 8t \cdot e^{-4t}$$

$$y \cdot e^{-4t} = \int \underbrace{8t}_{u} \cdot \underbrace{e^{-4t}}_{v'} \, dt$$

$$y \cdot e^{-4t} = \underbrace{8t}_{u} \cdot \underbrace{\left(-\tfrac{1}{4}\right) e^{-4t}}_{v} - \int \underbrace{8}_{u'} \cdot \underbrace{\left(-\tfrac{1}{4}\right) e^{-4t}}_{v} \, dt$$

$$y \cdot e^{-4t} = -2te^{-4t} + 2\int e^{-4t} \, dt$$

$$y \cdot e^{-4t} = -2te^{-4t} + 2 \cdot \left(-\tfrac{1}{4}\right) e^{-4t} + C \quad \Big| \cdot e^{4t}$$

$$\underline{\underline{y = -2t - \tfrac{1}{2} + Ce^{4t}}}$$

Control: $y' - 4y = -2 + Ce^{4t} \cdot 4 - 4\left(-2t - \tfrac{1}{2} + Ce^{4t}\right)$

$$= -2 + 4Ce^{4t} + 8t + 2 - 4Ce^{4t}$$

$$= 8t \quad \text{Correct!}$$

General solution if a and b are constants

$$y' + ay = b \quad | \cdot e^{at}$$

$$y' \cdot e^{at} + aye^{at} = be^{at}$$

$$(y \cdot e^{at})' = be^{at}$$

$$ye^{at} = \int be^{at}\,dt$$

$$y \cdot e^{at} = \frac{b}{a} e^{at} + C \quad | \cdot e^{-at}$$

$$\underline{y = \frac{b}{a} + C e^{-at}}$$

If $a(t)$ and $b(t)$ are not constants:

$$y' + a(t) \cdot y = b(t)$$

We must find an integrating factor u

$$\overbrace{u \cdot y' + \underbrace{u \cdot a(t)}_{u'} \cdot y}^{(u \cdot y)'} = u \cdot b(t)$$

$$\Rightarrow u' = u \cdot a(t)$$

We can see that $u = e^{\int a(t)\,dt}$ satisfies this condition.

$$\left(u' = e^{\int a(t)\,dt} \cdot \left(\int a(t)\,dt\right)' = e^{\int a(t)\,dt} \cdot a(t) = u \cdot a(t) \right)$$

Conclusion: Use $e^{\int a(t)\,dt}$ as integrating factor.

Ex : $* \quad y' + ty = 3t \qquad a(t) = t \ , b(t) = 3t$

Integrating factor:

$$e^{\int t\,dt} = e^{\frac{1}{2}t^2}$$

Multiply $*$ by $e^{\frac{1}{2}t^2}$

$$y' \cdot e^{\frac{1}{2}t^2} + tye^{\frac{1}{2}t^2} = 3t \cdot e^{\frac{1}{2}t^2}$$

$$\int \left(y \cdot e^{\frac{1}{2}t^2}\right)' dt = \int 3\underline{t} \cdot e^{\frac{1}{2}t^2} \underline{dt}$$

Let $v = \frac{1}{2}t^2$, $\frac{dv}{dt} = t$, $dv = t\,dt$

$$y \cdot e^{\frac{1}{2}t^2} = \int 3e^v \, dv$$

$$y \cdot e^{\frac{1}{2}t^2} = 3e^v + C$$

$$y \cdot e^{\frac{1}{2}t^2} = 3e^{\frac{1}{2}t^2} + C \qquad | \cdot e^{-\frac{1}{2}t^2}$$

$$\underline{\underline{y = 3 + C \cdot e^{-\frac{1}{2}t^2}}}$$

Summary:

If $y' + a(t) \cdot y = b(t)$ use $e^{\int a(t)\,dt}$ as an integrating factor

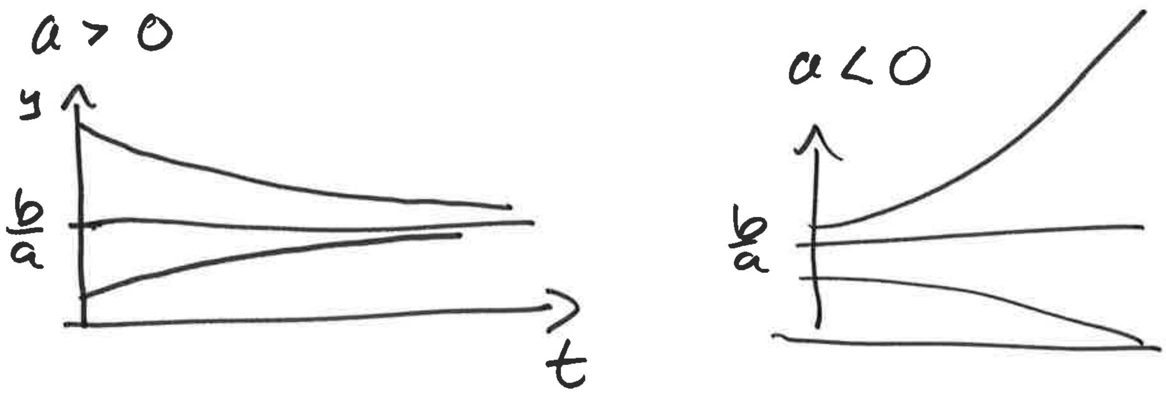
If $y' + ay = b$ then the solution is

$$\boxed{y = \frac{b}{a} + C \cdot e^{-at}}$$

$a > 0$ $\quad \lim_{t \to \infty} y = \frac{b}{a}$ (since $\lim_{t \to \infty} e^{-at} = 0$)

and the equation is call asymptotically stable

If $a < 0$ then $\lim_{t \to \infty} y = \pm\infty$



Ex. Supply and Demand

$$\boxed{\begin{aligned} D &= 5000 - 4p \\ S &= 1000 + 6p \\ p' &= 0{,}5\,(D - S) \end{aligned}}$$

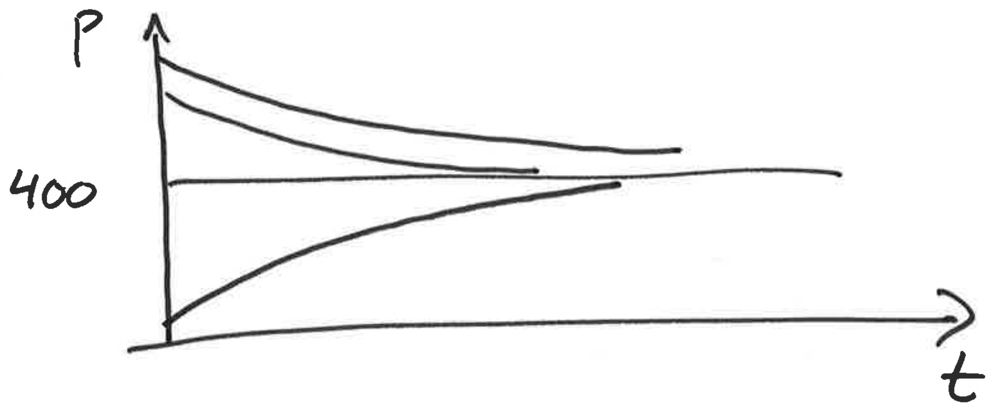
This gives

$$p' = 0{,}5\,(5000 - 4p - (1000 + 6p))$$

$$p' = 2000 - 5p$$

$$p' + 5p = 2000$$

$$p = \frac{2000}{5} + Ce^{-5t} \rightarrow 4000 \text{ when } t \to \infty$$



In general:

$$\boxed{\begin{aligned} D &= a - bp \\ S &= \alpha + \beta p \\ p' &= \lambda (D - S) \end{aligned}}$$

$$\Downarrow$$

$$p' = \lambda \left((a - bp - (\alpha + \beta p) \right)$$

$$p' = \lambda (a - \alpha) - \lambda (b + \beta) p \,)$$

$$p' + \lambda (b + \beta) p = \lambda (a - \alpha)$$

$$p = \frac{\lambda (a - \alpha)}{\lambda (b + \beta)} + c e^{-\lambda (b + \beta) t}$$

Exact differential equations.

ex.

$$\boxed{1 + t\cdot y^2 + t^2\cdot y\cdot y' = 0}$$

$$\Updownarrow$$

$$y' = \frac{-1 - ty^2}{t^2 y}$$

not separable
not linear.

An equation of the form:

$$\boxed{p(t,y)\cdot y' + q(t,y) = 0}$$

is called exact

$\Leftrightarrow$ $\exists$ $h(t,y)$ such that

$\Leftrightarrow$ if and only if; $\exists$ there exists

$$h'_y = p(t,y) \quad \text{and} \quad h'_t = q(t,y)$$

The solution is given by

$$h(t,y) = C$$

in the example:

$$\underbrace{t^2\cdot y}_{p(t,y)}\cdot y' + \underbrace{ty^2+1}_{q(t,y)} = 0$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$h'_y = t^2 y \qquad h'_t = ty^2 + 1$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\underbrace{h = \tfrac{1}{2}t^2y^2 + C(t) \qquad h = \tfrac{1}{2}t^2y^2 + t}$$

$$\Rightarrow h(t,y) = \tfrac{1}{2}t^2y^2 + t$$

and the equation is exact

solution:

$$h(t,y) = c$$

$$\frac{1}{2}t^2y^2 + t = c \qquad \text{implisitt}$$

$$\Updownarrow$$

$$y^2 = \frac{c-t}{\frac{1}{2}t^2} = \frac{2(c-t)}{t^2}$$

$$\underline{\underline{y = \pm\sqrt{\frac{2(c-t)}{t^2}}}} \qquad \text{general solution}$$

if we have the initial value:

$$y(1) = 1$$

then

$$1 = \sqrt{\frac{2(c-1)}{1}}$$

$$\Updownarrow$$

$$c = \frac{3}{2}$$

$$y = \sqrt{\frac{2(\frac{3}{2}-t)}{t^2}}$$

$$\underline{\underline{y = \sqrt{\frac{3-2t}{t^2}}}} \qquad \text{particular solution}$$