Note: How to sketch subsets of $\mathbb{R}^n$ and to determine if it is open/closed, convex, bounded.

① How to sketch a subset

> We only need a rough sketch

Ex: A $D = \{(x,y) : 1 \leq y \leq x^2+1\}$

the set of points (x,y) in $\mathbb{R}^2$ such that

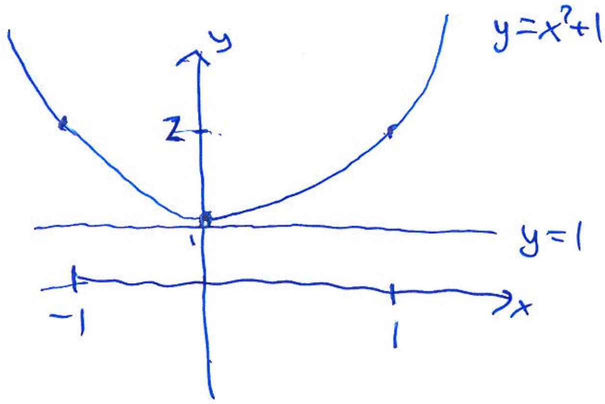
$$y \geq 1 \quad \text{and} \quad y \leq x^2+1$$

(that is, between $y=1$ and $y=x^2+1$).

a) Start with the boundary of D

In this case, look at $y=1$ and $y=x^2+1$.

> Replace inequalities ($\leq, \geq$) with equalities ($=$).

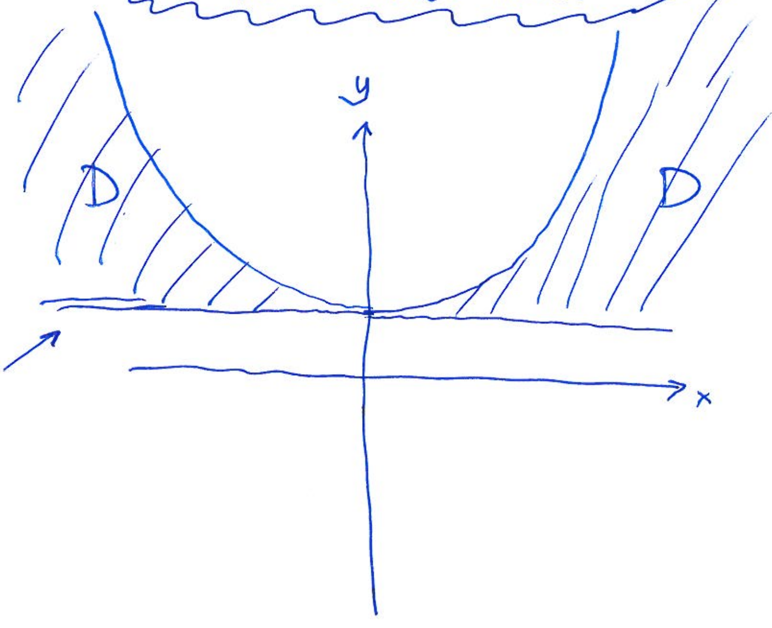


> When sketching a curve like $y=x^2+1$, it is easiest if you know what kind of curve it is ($y=x^2+1$ is a parabola with min. in the point $(0,1)$). If you don't recognize what kind of curve it is, plot some points on the curve to get a rough idea.

b) Sketch the set D

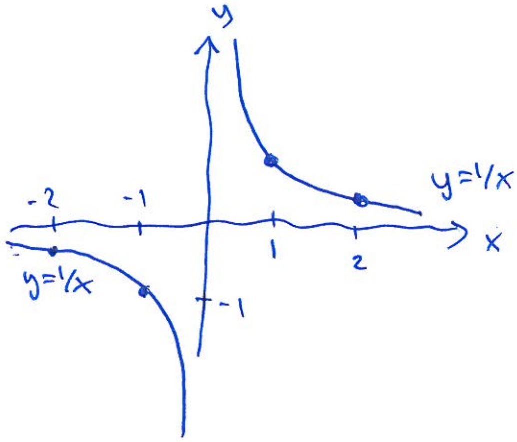
Look at $y \geq 1$ and $y \leq x^2+1$

The set D is the set of points between $y=1$ and $y=x^2+1$, and the graphs $y=1$, $y=x^2+1$ are included (since we have $\leq, \geq$).



Ex: B $D = \{(x,y) : xy \leq 1\}$
the set of points (x,y) in $\mathbb{R}^2$
such that $xy \leq 1$

a) Boundary given by $xy = 1$,
which is the same as $y = 1/x$.
This is a hyperbola.
If you don't recognize $y = 1/x$
as a hyperbola, plot some
points: $\left\{ \begin{array}{l} (1,1),\ (2,1/2) \\ (-1,-1),\ (-2,-1/2) \end{array} \right.$

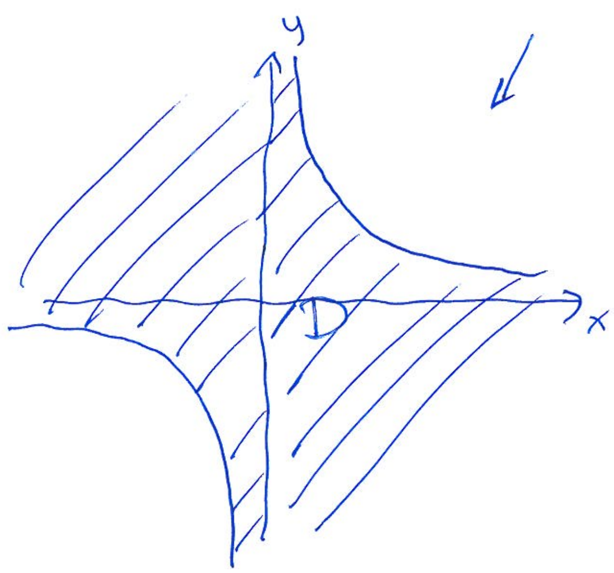


b) Sketch D: $xy \leq 1$
See that the boundary is part of
the set, since the condition is
given by $\leq$.

If $x>0$, then $\left.\begin{array}{c} xy \leq 1 \\ \Updownarrow \\ y \leq 1/x \end{array}\right\}$ For $x>0$, use the region under the curve $y=1/x$ (including the curve $y=1/x$)

If $x<0$, then $\left.\begin{array}{c} xy \leq 1 \\ y \geq 1/x \end{array}\right\}$ For $x<0$, use the region over the curve $y=1/x$ (including the curve $y=1/x$)

If $x=0$, then $xy = 0 \leq 1$ for all values of y, so use all of the y-axis $(x=0)$.



2) How to determine if a set is open/closed, convex, bounded

i) Open/closed:

Open $\Leftrightarrow$ no boundary points included in the set D
Closed $\Leftrightarrow$ all —— " ——

In practise, sets defined by

$$\boxed{\begin{array}{l} <, > \ : \ \text{open} \\ \leq, \geq, = \ : \ \text{closed} \end{array}}$$

We can usually see this by looking at the conditions (without drawing the set).
In some difficult cases, it may be necessary to draw the set and find its boundary.

ii) Convex:

Convex $\Leftrightarrow$ For any two points A, B in the set D, the line segment [A,B] lies inside the set D

With a sketch of D, we can usually see this graphically.

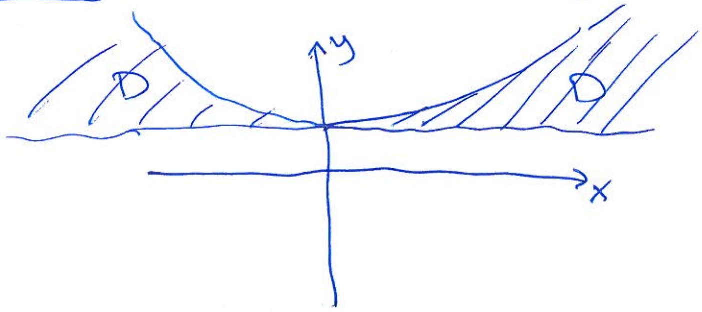
iii) Bounded:

Bounded $\Leftrightarrow$ There is a circle (in dim. 2) or a sphere (in higher dim.) with finite radius such that all points of D are inside the circle/sphere.

With a sketch of D, we can usually see this graphically.

It can be difficult to determine if a set D is convex/bounded directly from the conditions (without a drawing).

Ex. A: $D = \{(x,y) : 1 \le y \le x^2+1\}$

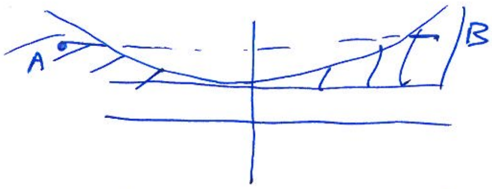


We have that:

i) D is closed, not open

(since the boundary of D, the graphs $y=1$ and $y = x^2+1$, are included in D; (i.e. $\le$)).

ii) D is not convex

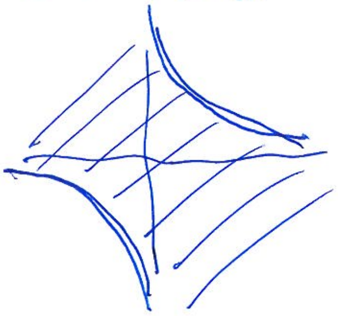


With A, B in D chosen as in the drawing, the line segment $[A,B]$ is not inside D

iii) D is not bounded

(since there are points in D infinitely large values of x, so there is no circle large enough to contain all pts. of D).

Ex. B: $D = \{(x,y) : xy \le 1\}$

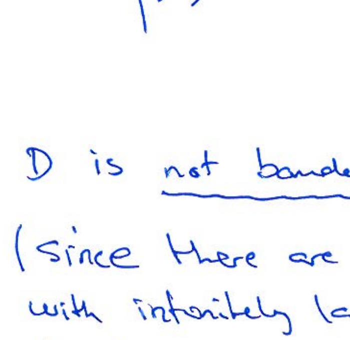


i) D is closed, not open

(since the boundary $xy=1$ is in D; ie. $\le$)

ii) D is not convex

(let $A = (1/2, 2)$ and $B = (2, 1/2)$, then set line segment $[A,B]$ goes outside of D)



This example is a bit difficult; the graph $y = 1/x$ must be drawn with the correct curvature to see that D is not convex.

iii) D is not bounded

(since there are pts. in D with infinitely large values of x)