Problem Session 3
GRA 6035 Mathematics

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BI Norwegian Business School

Solutions:

d) Kuhn - Tucker

i) max xy when $x+4y \le 16$ (std form)

$\mathcal{L} = xy - \lambda(x+4y)$

$\mathcal{L}'_x = y - \lambda = 0$

$\mathcal{L}'_y = x - \lambda \cdot 4 = 0$

Equality case : See Lagrange

Ineq. case: $\lambda = 0$, $x+4y < 16$
$x = y = 0, \lambda = 0$ $f = 0$

Best cand: $x = 8, y = 2 ; \lambda = 2$ from Lagrange
has $\lambda \ge 0$.

~~[illegible]~~

No max since if $x, y \to -\infty$ then $xy \to \infty$.
(but $x+4y \le 16$)

ii) max $x^2 y$ when $2x^2 + y^2 \le 3$ (std form)

Eq: $(\pm 1, 1; \frac{1}{2})$ best cand. $\lambda = 1/2 \ge 0$

Ineq: $2x^2 + y^2 < 3$ $\quad L'_x = 2xy - \lambda($[illegible]$) = 0$ $\quad 2xy = 0$
$\lambda = 0$ $\quad L'_y = x^2 - \lambda($[illegible]$) = 0$ $\quad x^2 = 0$ $\quad x = 0$

$\Rightarrow x = 0, -\sqrt{3} < y < \sqrt{3}, \lambda = 0 \Rightarrow f = 0$

NDCQ still holds. Bounded. $\Rightarrow$ Max = 1 for $(\pm 1, 1)$

min $x^2 y$ = max $-x^2 y$: Opposite signs of λ

Eq.: $(\pm 1, -1; -1/2)$ best. cand. $\lambda = -1/2 \le 0$ ok $f = -1$

Ineq: As above, $f = 0$

$\Rightarrow$ Min = -1 for $(\pm 1, -1)$

~~iii) max xyz when $\begin{cases} x^2 + y^2 \le 1 & g_1 \\ x + z \ge 1 \iff -x - z \le -1 & g_2 \end{cases}$~~ ~~Change sign of λ_2~~

~~Eq. g_1, g_2:~~ ~~[illegible]~~

iii) max xyz when $\begin{cases} x^2+y^2 \le 1 \\ x+z \ge 1 \end{cases}$

Std form:

$-x-z \le -1$

$$\boxed{\begin{array}{l} g_1 = x^2+y^2 \le 1 \\ g_2 = -x-z \le -1 \end{array}}$$

Candidates:

(a) $g_1=1,\ g_2=-1$: See Lagrange problem

Need solutions with $\lambda_1 \ge 0, \lambda_2 \le 0$

No such solutions

(b) $g_1=1,\ g_2<-1$: $\lambda_2=0$

$$\begin{cases} yz-\lambda_1\cdot 2x + \lambda_2\cdot(-1) = 0 \\ xz-\lambda_1\cdot 2y = 0 \\ xy + \lambda_2 = 0 \\ x^2+y^2 = 1 \end{cases}$$

$xy=0 \Rightarrow x=0, y=\pm 1$ or $x=\pm 1, y=0$

$\Downarrow$ $\lambda_1=0$, $z=0$; $\Downarrow$ $z=0$, $\lambda_1=0$

$(0,\pm1,0;0,0)$ $\underline{f=0}$; $(\pm1,0,0;0,0)$ $\underline{f=0}$ ok

(c) $g_1<1,\ g_2=-1$: $\lambda_1=0$

$$\begin{array}{l} yz-\lambda_1\cdot 2x+\lambda_2=0 \\ xz-\lambda_1\cdot 2y = 0 \\ xy + \lambda_2 = 0 \\ x+z = 1 \end{array}$$

$xz=0 \Rightarrow x=0, z=1$ or $z=0, x=1$

$\lambda_2=0$, $y=0$; $\lambda_2=0$, $y=0$

$(0,0,1;0,0)$ $f=0$; $(1,0,0;0,0)$ $f=0$ ok

(d) $g_1<1,\ g_2<-1$: $\lambda_1=\lambda_2=0$ $yz=xz=xy=0$

$\Downarrow$

$$\left.\begin{array}{l} x=0, y=0, z=\text{free} \\ x=0, y=\text{free}, z=0 \\ x=\text{free}, y=0, z=0 \end{array}\right\} f=0.$$

Best cand: Several cand. give $f=0$

This does not look like max. Maybe no max?

If $x = y = 1/2 \Rightarrow x^2 + y^2 = 1/4 + 1/4 = 1/2 < 1$ ok

$x + z \geq 1 \quad \Rightarrow z \geq 1 - x = 1 - 1/2 = 1/2$

This means $x = y = 1/2, z \geq 1/2$ are admissible

But $f = xyz = \frac{1}{4} z \to \infty$ as $z \to \infty$. No max

iv) See (Ex. 18.7)
v) " (Ex. 18.9)
vi) See (Prob. 18.12)

In each case the set is bounded, so the best candidate = computed point is the max.

③ Constrained optimalization problems.

Typical examples:

ⓐ max xy when $x+4y=16$ Equality constr.

ⓑ max xy when $x+4y\leq 16$ Ineq. constr.

Lagrenge problems: Equality constraints

max xy when $x+4y=16$ max = global max

i) Find candidates for max: F.O.C. + Constraint

$$\mathcal{L} = xy - \lambda\cdot(x+4y \quad)$$

$$\boxed{\begin{aligned} \mathcal{L}'_x &= y-\lambda\cdot 1 = 0 \\ \mathcal{L}'_y &= x-\lambda\cdot 4 = 0 \end{aligned}} \leftarrow \text{F.O.C.}$$

$$\boxed{x+4y=16} \leftarrow \text{Constraint}$$

$$\left.\begin{aligned} y&=\lambda \\ x&=4\lambda \end{aligned}\right\} \quad \begin{aligned} (4\lambda)+4(\lambda)&=16 \\ 8\lambda&=16 \\ \lambda&=2 \end{aligned} \quad \left.\begin{aligned} x&=8 \\ y&=2 \end{aligned}\right\} \quad \begin{aligned} &\text{One candidate:} \\ &(x,y;\lambda)=(8,2;2) \end{aligned}$$

ii) Check if $(x,y)=(8,2)$ is max.

If $\mathcal{L}(x,y;\lambda=2)$ is concave, then $(x,y)=(8,2)$ is a global max.

$\mathcal{L}(x,y;2) = xy - 2\cdot(x+4y) = xy - 2x - 8y$

$H(\mathcal{L}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad D_1 = 0$, $D_2 = -1$ $\quad$ not concave, (not convex)

If $x + 4y = 16$, then $x = 16 - 4y$

and

$xy = (16-4y)y = 16y - 4y^2 = g(y)$

$g'(y) = 16 - 8y = 0$

$\Rightarrow y = 2$

$g''(y) = -8$

This proves that $y=2$, $X=8$ is a <u>global max</u>.

If $(x^*, y^*, z^*; \lambda^*)$ satisfy FOC + Constraints in a Lagrange problem, then:

$$\boxed{\begin{array}{l} \text{If } \mathcal{L}(x,y,z;\lambda^*) \text{ is concave, then } (x^*,y^*,z^*) \text{ is max} \\ \text{If } \mathcal{L}(x,y,z;\lambda^*) \text{ is convex, then } (x^*,y^*,z^*) \text{ is min} \end{array}}$$

Solutions:

b) Lagrange problems:

i) See Extra Lecture 5.

iv) max x^2y when $2x^2+y^2=3$:

$$\boxed{f = x^2y}$$

$$\mathcal{L} = x^2y - \lambda(2x^2+y^2)$$

$$\boxed{\begin{array}{rl} \mathcal{L}'_x = & 2xy - \lambda\cdot 4x = 0 \\ \mathcal{L}'_y = & x^2 - \lambda\cdot 2y = 0 \\ & 2x^2+y^2=3 \end{array}}$$

$2x\cdot(y-2\lambda)=0 \Rightarrow \underline{x=0}$ or $\underline{y=2\lambda}$

$\underline{x=0}$: $\left.\begin{array}{l} y^2=3 \Rightarrow y=\pm\sqrt{3} \\ -\lambda\cdot 2y=0 \Rightarrow \lambda=0 \end{array}\right\}$ $\underline{(0,\pm\sqrt{3};0)\ \ f=0}$

$\underline{x\neq 0}$: $y=2\lambda$

$x^2=\lambda\cdot 2y = 2\lambda\cdot 2\lambda = 4\lambda^2$

$2x^2+y^2 = 2\cdot 4\lambda^2+(2\lambda)^2 = 12\lambda^2=3$

$\lambda^2 = 1/4 \quad \lambda = \pm 1/2$

$\lambda=1/2$: $y=1,\ x^2=1 \Rightarrow x=\pm 1 \Rightarrow \underline{(\pm 1, 1; \tfrac{1}{2})\ f=1}$

$\lambda=-1/2$: $y=-1,\ x^2=1 \Rightarrow x=\pm 1 \Rightarrow \underline{(\pm 1, -1; -\tfrac{1}{2})\ f=-1}$

Best candidate for max:

$(x,y)=(\pm 1, 1),\ \lambda=\frac{1}{2},\ f=1$

Check if max:

i) Look at $\mathcal{L}(x,y,\frac{1}{2}) = x^2y - \frac{1}{2}(2x^2+y^2) = x^2y - x^2 - \frac{1}{2}y^2$

$\mathcal{L}'_x = 2xy-2x$

$\mathcal{L}'_y = x^2 - y$

$$\mathcal{L}'' = \begin{pmatrix} 2y-2 & 2x \\ 2x & -1 \end{pmatrix}$$ not concave (not convex) $\Rightarrow$ no conclusion

ii) Look at admissible pts: $\{(x,y): 2x^2+y^2=3\}$ closed and bounded $\Rightarrow$ there is a max by extreme value thm.

Possible max:
- Cadidates computed above $(x,y)=(\pm 1, 1)\quad f=1$
- Special pts where NDCQ fails

NDCQ: $\text{rk}\ (\underset{\frac{\partial g}{\partial x}}{\underset{\uparrow}{4x}}\ \ \underset{\frac{\partial g}{\partial y}}{2y}) = 1$

NDCQ fails: $\text{rk}=0 \Updownarrow x=0, y=0$ impossible since $2x^2+y^2=3$

$\Downarrow$

Concl: $(x,y)=(\pm 1,1)$ gives $\underline{\text{max}}$ $\underline{\underline{f=1}}$

iii) max xyz when $\begin{cases} x^2+y^2=1 \\ x+z=1 \end{cases}$ $\boxed{f = xyz}$

$$\mathcal{L} = xyz - \lambda_1(x^2+y^2) - \lambda_2(x+z)$$

$$\mathcal{L}'_x = yz - \lambda_1 \cdot 2x - \lambda_2 = 0$$

$$\mathcal{L}'_y = xz - \lambda_1 \cdot 2y = 0 \quad \Rightarrow \lambda_1 = \frac{xz}{2y} \quad \text{or} \quad y=0$$

$$\mathcal{L}'_z = xy - \lambda_2 = 0 \quad \Rightarrow \lambda_2 = xy$$

$$x^2 + y^2 = 1$$

$$x + z = 1$$

Let's first check $y=0$: $x^2=1 \Rightarrow x = \pm 1$ $\left(\begin{matrix} x=1, z=0 \\ x=-1, z=2 \end{matrix}\right)$ $\lambda_2 = 0$ $x=0$ or $z=0$ $\Downarrow$ $x=-1, z=2$ impossible

$\Rightarrow$ $x=1, y=0, z=0, \lambda_2=0, \lambda_1=0$ $f=0$

Now, we may assume $y \neq 0$:

$\left.\begin{matrix} \lambda_1 = \frac{xz}{2y} \\ \lambda_2 = xy \end{matrix}\right\}$ put this into eqn. (1), (4), (5) to find x, y, z:

$$(1): \quad yz - \left(\frac{xz}{\cancel{2}y}\right)\cdot \cancel{2}x - (xy) = 0 \quad | \cdot y$$

$$y^2z - \cancel{x}\cdot x\cancel{z} - xy^2 = 0$$

$$y^2z - x^2z - xy^2 = 0$$

$\left.\begin{matrix} (4): \ y^2 = 1-x^2 \\ (5): \ z = 1-x \end{matrix}\right\}$ put this into eqn (1)

$$(1-x^2)\cdot(1-x) - x^2(1-x) - x(1-x^2) = 0$$

$$(1-x)\cdot\left[(1-x^2) - x^2 - x(1+x)\right] = 0$$

$$x=1 \quad \text{or} \quad 1-x^2-x^2-x-x^2=0$$

$$-3x^2 - x + 1 = 0$$

$$x = \frac{1 \pm \sqrt{1-4\cdot(-3)\cdot 1}}{2\cdot(-3)}$$

$$= \frac{1 \pm \sqrt{13}}{-6}$$

$x_1 = 1$, (gives $y=0$, same as above) $x_2 = \frac{1+\sqrt{13}}{-6} \simeq -0.77$, $x_3 = \frac{1-\sqrt{13}}{-6} \simeq 0.43$

Candidates:

x	y	z	λ_1	λ_2	$f=$
1	0	0	0	0	0
-0.77	± 0.64	1.77	∓ 1.06	∓ 0.49	∓ 0.87
0.43	± 0.90	0.57	± 0.14	± 0.39	± 0.22

Candidate for max: $\boxed{x = -0.73 \quad y = -0.64 \quad z = 1.77}$

Bounded set $\Rightarrow$ there is a max

NDCQ: $\text{rk}\begin{pmatrix} 2x & 2y & 0 \\ 1 & 0 & 1 \end{pmatrix} = 2$

NDCQ fails $\Leftrightarrow$ $\text{rk} \leq 1$ $\Leftrightarrow$ $\begin{vmatrix} 2x & 2y \\ 1 & 0 \end{vmatrix} = -2y = 0$ $\Leftrightarrow$ $x=y=0$ impossible

$\begin{vmatrix} 2x & 0 \\ 1 & 1 \end{vmatrix} = 2x = 0$

$\begin{vmatrix} 2y & 0 \\ 0 & 1 \end{vmatrix} = 2y = 0$

$\Downarrow$

Max: $f = 0.87$ when $x = -0.77$, $y = -0.64$, $z = 1.77$

iv) min x^2+y^2 when $x^2+xy+y^2=3$:

$\mathcal{L} = x^2+y^2 - \lambda(x^2+xy+y^2)$

$\mathcal{L}'_x = 2x - \lambda \cdot 2x - \lambda \cdot y = 0$ $\qquad 2x - \lambda(2x+y) = 0$

$\mathcal{L}'_y = 2y - \lambda \cdot (x+2y) = 0$ $\qquad 2y - \lambda(x+2y) = 0$

$x^2+xy+y^2 = 3$

$\lambda = 0$: $x=y=0$ $\Rightarrow$ imp. $\Rightarrow \boxed{\lambda \neq 0}$

$y = \frac{1}{\lambda} \cdot 2x(1-\lambda) = 2x \cdot \frac{1-\lambda}{\lambda}$

$2 \cdot 2x \cdot \frac{1-\lambda}{\lambda} - \lambda \cdot \left(x + 4x\, \frac{1-\lambda}{\lambda}\right) = 0$

$4x \cdot \frac{1-\lambda}{\lambda} - x\lambda - 4x\lambda \frac{1-\lambda}{\lambda} = 0$

$x = 0$ or $4\,\frac{1-\lambda}{\lambda} - \lambda - 4\lambda \cdot \frac{1-\lambda}{\lambda} = 0 \quad | \cdot \lambda$

$y = \pm\sqrt{3}$

$\lambda = 0$

$\Downarrow$

imp.

$4(1-\lambda) - \lambda^2 - 4\lambda(1-\lambda) = 0$

$3\lambda^2 - 8\lambda + 4 = 0$

$\lambda = \frac{8 \pm \sqrt{64 - 4 \cdot 3 \cdot 4}}{2 \cdot 3} = \frac{8 \pm 4}{6} = 2, 2/3$

$\lambda = 2$: $-2x - 2y = 0 \Rightarrow x = -y \Rightarrow x^2 = 3$

$x = \pm\sqrt{3}$, $y = \mp\sqrt{3}$, $\lambda = 2$

$f = 6$

$\lambda = 2/3$: $\frac{2}{3}x - \frac{2}{3}y = 0 \Rightarrow x = y \Rightarrow 3x^2 = 3$

$x = \pm 1$, $y = \pm 1$, $\lambda = 2/3$

$f = 2$

Best cand. for min:

$(\pm 1, \pm 1; 2/3)$

$\mathcal{L}(x,y;2/3) = x^2+y^2 - \frac{2}{3}(x^2+xy+y^2)$

$= \frac{1}{3}x^2 + \frac{1}{3}y^2 - \frac{2}{3}xy$

$\mathcal{L}'' = \begin{pmatrix} 2/3 & -2/3 \\ -2/3 & 2/3 \end{pmatrix}$ $\qquad \Delta_1 = 2/3, 2/3$

$\Delta_2 = 0$

$\Downarrow$

convex

$\Downarrow$

$(\pm 1, \pm 1; 2/3)$ is min

v) See solution to (18.5)
vi) — " — (18.6)
vii) — " — (18.7)
viii) — " — (18.9) with $z=3$.

Solutions from [Simon, Blume] - See next pages.

In vi), vii), viii) the set is closed and bonded, so the computed points are max/min as indicated.

In v) the set is not bonded. Using the Lagrangian

$$L = x^2+y^2+z^2 - \lambda_1(3x+y+z-5) - \lambda_2(x+y+z-1)$$

we see that it is convex function in (x,y,z) for any λ_1, λ_2, so the computed pt. is min.

Notice the similarities and differences between the statement of Theorem 18.2 which treats equality constraints and the statement of Theorem 18.3 which covers inequality constraints:

(1) Both use the same Lagrangian L and both require that the derivatives of L with respect to the x_i's be zero.

(2) The condition that $\partial L/\partial \mu = h(x, y) - c = 0$ for equality constraints may no longer hold for inequality constraints since the constraint need not be binding at the maximizer in the inequality constraint case. It is replaced by two conditions:

$$\lambda \cdot [g(x, y) - b] = 0 \quad \text{and} \quad \frac{\partial L}{\partial \lambda} = g(x, y) - b \leq 0.$$

The second of these two conditions is simply a repetition of the inequality constraint itself.

(3) Both situations require that we check a constraint qualification. However, we need only check the constraint qualification for an inequality constraint if that constraint is binding at the solution candidate.

(4) There were no restrictions on the sign of the multiplier in the equality constraint situation; however, the multiplier for inequality constraints must be nonnegative.

(5) For equality constraints (and for problems with no constraints), the same first order conditions that work for maximization problems also hold for minimization problems. However, the argument, summarized in Figure 18.4, that $\nabla f(\mathbf{p})$ and $\nabla g(\mathbf{p})$ point in the *same* direction for inequality constraints holds only for the maximization problem. The same line of reasoning concludes that $\nabla f(\mathbf{p})$ and $\nabla g(\mathbf{p})$ must point in *opposite* directions in a constrained minimization problem. We will say more about the distinction between minimization problems and maximization problems in Section 18.5.

Example 18.7 Consider the problem of maximizing $f(x, y) = xy$ on the constraint set $g(x, y) = x^2 + y^2 \leq 1$. The only critical point of g occurs at the origin — far away from the boundary of the constraint set $x^2 + y^2 = 1$. So, the constraint qualification will be satisfied at any candidate for a solution. Form the Lagrangian

$$L(x, y, \lambda) = xy - \lambda(x^2 + y^2 - 1),$$

and write out the first order conditions described in Theorem 18.3:

$$\frac{\partial L}{\partial x} = y - 2\lambda x = 0, \qquad \frac{\partial L}{\partial y} = x - 2\lambda y = 0,$$

$$\lambda(x^2 + y^2 - 1) = 0, \quad x^2 + y^2 \leq 1, \qquad \lambda \geq 0.$$

The first two equations yield

$$\lambda = \frac{y}{2x} = \frac{x}{2y}, \quad \text{or} \quad x^2 = y^2. \tag{18}$$

If $\lambda = 0$, then $x = y = 0$. This combination satisfies all the first order conditions, so it is a candidate for a solution. If $\lambda \neq 0$, then the third equation becomes $x^2+y^2-1 = 0$. Combining this with (18), we find that $x^2 = y^2 = 1/2$, or $x = \pm 1/\sqrt{2}$, $y = \pm 1/\sqrt{2}$. Combining these with the equation for λ in (18), we find the following four candidates:

$$\begin{aligned}
x &= +\frac{1}{\sqrt{2}}, & y &= +\frac{1}{\sqrt{2}}, & \lambda &= +\frac{1}{2};\\
x &= -\frac{1}{\sqrt{2}}, & y &= -\frac{1}{\sqrt{2}}, & \lambda &= +\frac{1}{2};\\
x &= +\frac{1}{\sqrt{2}}, & y &= -\frac{1}{\sqrt{2}}, & \lambda &= -\frac{1}{2};\\
x &= -\frac{1}{\sqrt{2}}, & y &= +\frac{1}{\sqrt{2}}, & \lambda &= -\frac{1}{2}.
\end{aligned}$$

We disregard the last two candidates since they involve a negative multiplier. So, including (0, 0, 0), there are three candidates which satisfy all five first order conditions. Plugging these three into the objective function, we find that

$$x = \frac{1}{\sqrt{2}}, \; y = \frac{1}{\sqrt{2}} \quad \text{and} \quad x = -\frac{1}{\sqrt{2}}, \; y = -\frac{1}{\sqrt{2}}$$

are the solutions of our original problem.

The two points with the negative multipliers are the solutions of the problem of *minimizing* xy on the constraint set $x^2 + y^2 \leq 1$.

One way to think of condition c in Theorem 18.3 is that if $\lambda > 0$, we know the constraint will be binding and we can treat it as an equality constraint instead of as an inequality constraint — a much simpler criterion to work with. In some economics problems this type of analysis can give us useful information about the phenomenon under study, as the following example illustrates.

Example 18.8 Consider once again the standard utility maximization problem of Example 18.1. We continue to ignore the nonnegativity constraints but now do not force the budget constraint to be binding in the statement of the problem. We will see that the tightness of the budget constraint — the conclusion that the consumer spends all the available income — is a consequence of a natural monotonicity assumption on the utility function.

For ease of notation, assume that the first k_0 constraints are binding at $\mathbf{x}^*$ and that the last $k - k_0$ constraints are not binding. Suppose that the following nondegenerate constraint qualification is satisfied at $\mathbf{x}^*$.

The rank at $\mathbf{x}^*$ of the Jacobian matrix of the *binding* constraints

$$\begin{pmatrix} \frac{\partial g_1}{\partial x_1}(\mathbf{x}^*) & \cdots & \frac{\partial g_1}{\partial x_n}(\mathbf{x}^*) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_{k_0}}{\partial x_1}(\mathbf{x}^*) & \cdots & \frac{\partial g_{k_0}}{\partial x_n}(\mathbf{x}^*) \end{pmatrix}$$

is k_0 — as large as it can be.

Form the Lagrangian

$$L(x_1, \dots, x_n, \lambda_1, \dots, \lambda_k) \equiv f(\mathbf{x}) - \lambda_1[g_1(\mathbf{x}) - b_1] - \cdots - \lambda_k[g_k(\mathbf{x}) - b_k].$$

Then, there exist multipliers $\lambda_1^*, \dots, \lambda_k^*$ such that:

$$\begin{aligned} &(a) && \frac{\partial L}{\partial x_1}(\mathbf{x}^*, \lambda^*) = 0, \dots, \frac{\partial L}{\partial x_n}(\mathbf{x}^*, \lambda^*) = 0, \\ &(b) && \lambda_1^*[g_1(\mathbf{x}^*) - b_1] = 0, \dots, \lambda_k^*[g_k(\mathbf{x}^*) - b_k] = 0, \\ &(c) && \lambda_1^* \geq 0, \dots, \lambda_k^* \geq 0, \\ &(d) && g_1(\mathbf{x}^*) \leq b_1, \dots, g_k(\mathbf{x}^*) \leq b_k. \end{aligned}$$

Remark The constraint qualification in the statement of Theorem 18.4 is the natural generalization of the constraint qualifications in Theorems 18.2 and 18.3. This condition involves only the *binding* constraints since the nonbinding constraints should play no role in the first order conditions. Then, we treat the binding constraints just as we did the equality constraints in Theorem 18.2, by assuming that their Jacobian has maximal rank. We will still abbreviate this version of the nondegenerate constraint qualifications as NDCQ.

Example 18.9 Consider the problem of maximizing $f(x, y, z) = xyz$ on the constraint set defined by the inequalities

$$x + y + z \leq 1, \quad x \geq 0, \quad y \geq 0, \quad \text{and} \quad z \geq 0.$$

This is the typical example of a utility maximization problem in a three-dimensional commodity space. Since we need to write all our inequality constraints consistently — with a $\leq$ — we write the three nonnegativity constraints as

$$-x \leq 0, \quad -y \leq 0, \quad \text{and} \quad -z \leq 0.$$

The Jacobian of the constraint functions is

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Since its columns are linearly independent, it has rank three. Since at most three of the four constraints can be binding at any one time, the NDCQ holds at any solution candidate. Form the Lagrangian

$$L(x, y, z, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = xyz - \lambda_1(x + y + z - 1) \\ - \lambda_2(-x) - \lambda_3(-y) - \lambda_4(-z).$$

Because of the double minus signs in the last three terms of this Lagrangian, we can rewrite it more aesthetically as

$$L(x, y, z, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = xyz - \lambda_1(x + y + z - 1) + \lambda_2 x + \lambda_3 y + \lambda_4 z.$$

From now on, we will treat nonnegativity constraints this way, by including them in the Lagrangian as $+\lambda_i x_i$ rather than as $-\lambda_i(-x_i)$. We now write out the complete set of first order conditions, according to Theorem 18.4:

$$(1) \quad \frac{\partial L}{\partial x} = yz - \lambda_1 + \lambda_2 = 0,$$

$$(2) \quad \frac{\partial L}{\partial y} = xz - \lambda_1 + \lambda_3 = 0,$$

$$(3) \quad \frac{\partial L}{\partial z} = xy - \lambda_1 + \lambda_4 = 0,$$

(4) $\lambda_1(x + y + z - 1) = 0,$ (5) $\lambda_2 x = 0,$

(6) $\lambda_3 y = 0,$ (7) $\lambda_4 z = 0,$

(8) $\lambda_1 \geq 0,$ (9) $\lambda_2 \geq 0,$

(10) $\lambda_3 \geq 0,$ (11) $\lambda_4 \geq 0,$

(12) $x + y + z \leq 1,$ (13) $x \geq 0,$

(14) $y \geq 0,$ (15) $z \geq 0.$

Rewrite conditions 1, 2, and 3, without minus signs, as

$$\lambda_1 = yz + \lambda_2 = xz + \lambda_3 = xy + \lambda_4. \tag{20}$$

We will look at two cases: $\lambda_1 = 0$ and $\lambda_1 > 0$.

If $\lambda_1 = 0$ in equation (20), then because every variable in equation (20) is nonnegative,

$$yz = xz = xy = 0 \quad \text{and} \quad \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0. \tag{21}$$

Equations (21) lead to the (infinite) set of solution candidates in which two of the variables equal zero and the third is any number in the interval [0, 1]. In particular, the objective function equals zero for all (x, y, z) which satisfy equations (21).

Next, look at the case $\lambda_1 > 0$. By condition 4, $x+y+z = 1$; at least one of x, y, z must be nonzero. Suppose for a moment that $x = 0$. Then, using equations (20) and the assumption that $\lambda_1 > 0$, we see that $\lambda_3 = \lambda_4 = \lambda_1 > 0$. But then, conditions 6 and 7 imply that $y = z = 0$ — a contradiction to $x + y + z = 1$. Since the assumption that $x = 0$ leads to a contradiction, we conclude that $x > 0$. Similar arguments show that y and z are positive too. Then, conditions 5, 6, and 7 imply that $\lambda_2 = \lambda_3 = \lambda_4 = 0$ and equations (20) become simply

$$yz = xz = xy.$$

It follows now that

$$x = y = z = \frac{1}{3} \tag{22}$$

and, by using equation (20) once more, that $\lambda_1 = 1/9$. Since

$$f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{27} > 0,$$

(22) is the solution of the constrained maximization problem.

As this example shows, the solution of a constrained maximization problem usually involves breaking the first order conditions into a number of cases. It is often easiest to start with the nonnegativity constraints or the signs of the multipliers. In Example 18.9, we first worked with the case $\lambda_1 = 0$. Each case needs to be carried out until either a complete candidate for a solution is computed, including values for the multipliers, or a contradiction to one of the first order conditions is reached. While working with any given case, one might have to break that case into two subcases depending on whether or not a second inequality constraint is binding or not. In Example 18.9, while studying the case $\lambda_1 > 0$, we had to examine two subcases depending on the sign of x.

In economic theory, however, one rarely needs to compute the maxima or minima of a specific problem. One is usually more interested in studying the first order conditions which arise in a specific *type* of problem, since these can lead to interesting relationships between the variables of the problem or even to

Solving for λ and substituting gives $2x^3 - x - 2 = 0$. Thus $x \approx 1.165$, and so $y = x^2 \approx 1.357$. Finally $\nabla(y - x^2) = (-2x, 1) \neq (0, 0)$ so NDCQ holds.

18.4 The location and type of the critical points are independent of $k > 0$, so assume without loss of generality that $k = 1$.

$$\begin{aligned} \max \quad & x_1^a x_2^{1-a} \\ \text{subject to} \quad & (p_1 x_1 + p_2 x_2 - I) = 0. \end{aligned}$$

The Lagrangian is

$$L = x_1^a x_2^{1-a} - \lambda(p_1 x_1 + p_2 x_2 - I).$$

The first order conditions are

$$\begin{aligned} L_{x_1} &= a x_1^{a-1} x_2^{1-a} - \lambda p_1 = 0 \\ L_{x_2} &= (1 - a) x_1^a x_2^{-a} - \lambda p_2 = 0 \\ L_\lambda &= p_1 x_1 + p_2 x_2 - I = 0. \end{aligned}$$

The solution is

$$x_1 = \frac{aI}{p_1} \qquad x_2 = \frac{(1-a)I}{p_2}.$$

Since the constraint is linear, NDCQ holds.

18.5

$$\begin{aligned} \min \quad & x^2 + y^2 + z^2 \\ \text{subject to} \quad & 3x + y + z = 5 \\ & x + y + z = 1. \end{aligned}$$

The Lagrangian is

$$L = x^2 + y^2 + z^2 - \lambda_1(3x + y + z - 5) - \lambda_2(x + yz - 1).$$

The first order conditions are

$$\begin{aligned} L_x &= 2x - 3\lambda_1 - \lambda_2 = 0 \\ L_y &= 2y - \lambda_1 - \lambda_2 = 0 \\ L_z &= 2z - \lambda_1 - \lambda_2 = 0 \end{aligned}$$

$$\begin{aligned} L_{\lambda_1} &= 3x + y + z - 5 = 0 \\ L_{\lambda_2} &= x + y + z - 1 = 0. \end{aligned}$$

This linear system of five equations in five unknowns has a unique solution: $(2, -1/2, -1/2)$. The Jacobian of the constraints is $\begin{pmatrix} 3 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, which has rank 2, and so the NDCQ holds.

18.6 Substitute $y = 0$ into all the equations.

$$\begin{aligned} \text{max} \quad (\text{min}) \quad & x + z^2 \\ \text{subject to} \quad & x^2 + z^2 = 1. \end{aligned}$$

The Lagrangian is

$$L = x + z^2 - \lambda(x^2 + z^2 - 1)$$

and the first order conditions are

$$\begin{aligned} L_x &= 1 - 2\lambda x = 0 \\ L_z &= 2z - \lambda 2z = (1 - \lambda)2z = 0 \\ L_\lambda &= x^2 + z^2 - 1 = 0. \end{aligned}$$

There are four solutions:

$$(x, y, z, \lambda) = \begin{cases} (1/2, 0, \sqrt{3}/2, 1) \\ (1/2, 0, -\sqrt{3}/2, 1) \\ (1, 0, 0, 1/2) \\ (-1, 0, 0, -1/2) \end{cases}.$$

A check shows that the first two correspond to local maxima with a value of $5/4$, the third to a critical point with a value of 1, and the last to a local minimum with a value of -1.

18.7 Substitute the constraint $xz = 3$ into the objective function:

$$\begin{aligned} \text{max} \quad & 3 + yz \\ \text{subject to} \quad & y^2 + z^2 = 1. \end{aligned}$$

The Lagrangian is

$$L = 3 + yz - \lambda(y^2 + z^2 - 1).$$

The first order conditions are

$$\begin{aligned} L_y &= z - 2\lambda y = 0 \\ L_z &= y - 2\lambda z = 0 \\ L_\lambda &= y^2 + z^2 - 1 = 0. \end{aligned}$$

The solutions are the four (y, z) pairs such that $y = \pm 1/\sqrt{2}$ and $z = \pm 1/\sqrt{2}$. For a maximum, y and z must have the same sign, so the solutions are $(3\sqrt{2}, 1/\sqrt{2}, 1/\sqrt{2})$ and $(-3\sqrt{2}, -1/\sqrt{2}, -1/\sqrt{2})$. The value of the maximand in each case is $7/2$, and NDCQ holds.

18.8 Suppose there are n variables and m constraints. The Jacobian of the constraints is

$$\begin{pmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_m}{\partial x_1} & \cdots & \frac{\partial h_m}{\partial x_n} \end{pmatrix}.$$

This matrix can have rank m only if $m \leq n$.

18.9 A simple substitution makes this much easier. Let $X = x^2$, $Y = y^2$ and $Z = z^2$. The maximization problem is now

$$\begin{aligned} &\max \quad XYZ \\ &\text{subject to} \quad X + Y + Z = c^2 \end{aligned}$$

together with inequality constraints $X \geq 0$, $Y \geq 0$ and $Z \geq 0$, which we will ignore for the moment. This is a familiar problem. The Lagrangian is

$$L = XYZ - \lambda(X + Y + Z - c^2).$$

The first order conditions are

$$\begin{aligned} L_X &= YZ - \lambda = 0 \\ L_Y &= XZ - \lambda = 0 \\ L_Z &= XY - \lambda = 0 \\ L_\lambda &= X + Y + Z - c^2 = 0 \end{aligned}$$

The solution is

$$X = Y = Z = \frac{c^2}{3}$$

and we see that the inequality constraints are never binding. Going back to the original problem, the solution is

$$x = y = z = \frac{c}{\sqrt{3}}.$$

The value of the objective function at its maximum is $(c^2/3)^3$, so

$$(c^2/3)^3 \geq \left(\frac{x^2 + y^2 + z^2}{3}\right)^3$$

for all (x, y, z) in the constraint set. Ranging over all values of c,

$$(x^2y^2z^2)^{1/3} \leq \left(\frac{x^2 + y^2 + z^2}{3}\right).$$

18.10 The Lagrangian is

$$L = x^2 + y^2 - \lambda(2x + y - 2) + \nu_1 x + \nu_2 y.$$

The first order conditions are

$$\begin{aligned} L_x &= 2x - 2\lambda + \nu_1 = 0 \\ L_y &= 2y - \lambda + \nu_2 = 0 \\ &\lambda(2x + y - 2) = 0 \\ &\nu_1 x = 0 \\ &\nu_2 y = 0 \\ &\nu_1 \geq 0, \quad \nu_2 \geq 0, \quad \lambda \geq 0. \end{aligned}$$

Solve by enumerating cases. Is there a solution with $x = 0$? If so, then $\nu_1 = 2\lambda$ and $y = 2$. If $y = 2$, then $\nu_2 = 0$, so $\lambda = 4$ and $\nu_1 = 8$, which is consistent with the FOCs. This is a solution. Is there a solution with $y = 0$? If so, then $\nu_2 = \lambda$ and $x = 1$. If $x = 1$, then $\nu_1 = 0$, so $\lambda = 1$ and $\nu_1 = 1$, which is consistent with the FOCs. This is a solution. $x = y = \nu_1 = \nu_2 = \lambda = 0$ is a solution. If neither x nor y are 0, then $\nu_1 = \nu_2 = 0$. Then $x = 4/5$ and so $y = 2/5$. This is consistent. Among these four points the global maximum occurs at $(0, 2)$ and the value of f is 4.

18.11 The Lagrangian is

$$L = 2y^2 - x - \lambda(x^2 + y^2 - 1) + \nu_1 x + \nu_2 y.$$

The first order conditions are

$$\begin{aligned} L_x &= -1 - 2\lambda x + \nu_1 = 0 \\ L_y &= 4y - 2\lambda y + \nu_2 = 0 \\ &\lambda(x^2 + y^2 - 1) = 0 \\ &\nu_1 x = 0 \\ &\nu_2 y = 0 \\ &\nu_1 \geq 0, \quad \nu_2 \geq 0, \quad \lambda \geq 0. \end{aligned}$$

The only solution to the first order conditions is

$$x = 0 \quad y = 1 \quad \nu_1 = 1 \quad \nu_2 = 0 \quad \lambda = 2$$

so the optimum is $x = 0$ and $y = 1$, and the value of f is 2.

18.12 *a*) The problem is

$$\begin{aligned} \max \quad & xyz + z \\ \text{subject to} \quad & x^2 + y^2 + z \leq 6 \\ & x \geq 0, y \geq 0, z \geq 0. \end{aligned}$$

The first order conditions are

$$\begin{aligned} L_x &= yz - 2\lambda x + \nu_1 = 0 \\ L_y &= xz - 2\lambda y + \nu_2 = 0 \\ L_z &= xy + 1 - \lambda + \nu_3 = 0 \\ &\lambda(x^2 + y^2 + z - 6) = 0 \\ &\nu_1 x = 0 \\ &\nu_2 y = 0 \\ &\nu_3 z = 0 \\ &\nu_1 \geq 0, \quad \nu_2 \geq 0, \quad \nu_3 \geq 0, \quad \lambda \geq 0. \end{aligned}$$

b) There is no solution to the first order conditions with $\lambda = 0$, because $\lambda = 0$ implies $xy + 1 + \nu_3 = 0$, and this equation has no nonnegative solution. Since $\lambda > 0$, the constraint must be binding at every solution to the first order conditions.

c) If $x = 0$, then $\nu_2 = 2\lambda y$. Since $\lambda > 0$, $\nu_2 y = 2\lambda y^2 = 0$ implies $y = 0$ and $\nu_2 = 0$. Therefore $z = 6$, $\nu_1 = 0$, $\nu_3 = \lambda - 1$, and $\lambda \geq 1$.

d) If $x > 0$, then $\nu_1 = 0$ so $yz = 2\lambda x$. Since $\lambda > 0$, y and z are both positive, so ν_2 and ν_3 are both 0. Thus four equations in four unknowns are the first order conditions for L_x, L_y and L_z and the constraint equality:

$$\begin{aligned} yz - 2\lambda x &= 0 \\ xz - 2\lambda y &= 0 \\ xy + 1 - \lambda &= 0 \\ x^2 + y^2 + z &= 6. \end{aligned}$$

Solving for λ and substituting gives the equation system

$$\begin{aligned} -2x - 2x^2y + yz &= 0 \\ -2y - 2xy^2 + xz &= 0 \\ x^2 + y^2 + z &= 6. \end{aligned}$$

e) This equation system has only one solution that satisfies all the nonnegativity constraints: $x = 1, y = 1$, and $z = 4$. Then $\lambda = 2$.

18.13 The Lagrangian is

$$L = U(x_1, x_2) - \lambda(p_1x_1 + p_2x_2 - I) + \nu_1x_1 + \nu_2x_2.$$

The first order conditions include

$$\begin{aligned} L_{x_1} &= U_{x_1} - \lambda p_1 + \nu_1 = 0 \\ L_{x_2} &= U_{x_2} - \lambda p_2 + \nu_2 = 0 \\ &\lambda(p_1x_1 + p_2x_2 - I) = 0. \end{aligned}$$

If $\lambda = 0$, then $U_{x_i} + \nu_i = 0$ has to have a nonnegative solution, and this is impossible if at every $(x_1, x_2) \geq 0$ at least one of the U_{x_i} exceeds 0. Thus $\lambda > 0$, so $p_1x_1 + p_2x_2 = I$.

At most one nonnegativity constraint can bind because the origin cannot solve the first order conditions (since $\lambda > 0$). Thus there are at most two binding constraints: The budget constraint and one of the inequality constraints. If one inequality constraint binds, the budget-constraint row of the matrix of derivatives of the binding constraints has two nonzero entries, and the row corresponding to the inequality constraint has one 0 and one 1, so the matrix is nonsingular. If only the budget constraint binds, the positivity of any one price guarantees that the matrix (now 1 by 2) has full rank. In either case NDCQ holds.

$$\lambda = \frac{2x}{(2x + y)} = \frac{2y}{(x + 2y)}.$$

Consequently $x = \pm y$. The solutions to the first order conditions are $(x, y) = \pm(\sqrt{1.1}, -\sqrt{1.1})$ with $\lambda = 2/3$ and $(x, y) = \pm(\sqrt{3.3}, -\sqrt{3.3})$ with $\lambda = 2$. Min square of distance is $1.1 + 1.1 = 2.2$; Max square of distance is $3.3 + 3.3 = 6.6$.

b) Using Theorem 19.1 and Exercise 18.2,

$$\begin{aligned} V(3.3) &\approx V(3) + \lambda^*(0.3) \\ &= 2 + (2/3)0.3 \quad = 2.2 \\ \text{or} \quad V(3.3) &= 6 + 2(.3) \quad\quad = 6.6, \end{aligned}$$

the exact same answers as in *a*.

19.2 *a*) $V(.8) \approx V(1) + \lambda^*(-.2) = 5/4 + 1(-.2) = 1.05.$

b) Using the method of Exercise 18.6,

$$\begin{aligned} &L_x = 1 - 2\lambda x = 0, \\ &L_y = 1 - 2\lambda y - \mu = 0, \\ &L_z = (1 - \lambda)2z = 0. \\ &\lambda = 1 \Longrightarrow x = 1/2, y = 0, z^2 = 0.8 - 0.25 = 0.55; z = \sqrt{0.55} \\ &x^2 + y + z^2 = 0.5 + 0 + 0.55 = 1.05, \quad \text{as in } a. \end{aligned}$$

19.3 *a*)

$$\begin{aligned} L &= 50x^{1/2}y^2 - \lambda(x + y - 80). \\ L_x &= 25x^{-1/2}y^2 - \lambda = 0, \\ L_y &= 100x^{1/2}y - \lambda = 0. \end{aligned}$$

Dividing,

$$\begin{aligned} 4x/y = 1 \quad &\Longrightarrow \quad y = 4x \\ &\Longrightarrow \quad x = 16, \quad y = 64, \quad Q = 819{,}200, \quad \lambda = 25{,}600. \end{aligned}$$

b) $Q^*(79) \approx Q^*(80) + \lambda^*(-1) = 819{,}200 - 25{,}600 = 793{,}600.$

c) As in *a*, $y = 4x$. $5x = 79 \Longrightarrow x = 15.8, y = 63.2.$

$Q = 50 \cdot (15.8)^{1/2}(63.2)^2 = 793{,}839.5.$

$\Delta Q = -25{,}360.5.$