Problems

1. Write down the Kuhn-Tucker conditions and solve them in the following Kuhn-Tucker problems:

a) $\max f(x,y) = xy$ subject to $x+4y \leq 16$
b) $\max f(x,y) = x^2y$ subject to $2x^2+y^2 \leq 3$
c) $\max f(x,y,z) = xyz$ subject to $x^2+y^2 \leq 1$ and $x+z \geq 1$
d) $\max f(x,y) = xy$ subject to $x^2+y^2 \leq 1$
e) $\max f(x,y,z) = xyz$ subject to $x+y+z \leq 1$, $x \geq 0$, $y \geq 0$ and $z \geq 0$
f) $\max f(x,y,z) = xyz+z$ subject to $x^2+y^2+z \leq 6$, $x \geq 0$, $y \geq 0$ and $z \geq 0$

2. Solve the Kuhn-Tucker problems in Problem 1.

3. When x thousand dollars is spent on labor and y thousand dollars is spent on equipment, a certain factory produces $Q(x,y) = 50x^{1/2}y^2$ units of output.

a) How should \$80.000 be allocated between labor and equipment to yield the largest possible output?
b) Use an envelope theorem to estimate the change in maximum output if this allocation is decreased by \$1.000.
c) Compute the exact change in b).

4. Write down the Lagrange conditions and solve them in the following Lagrange problems:

a) $\max f(x,y) = xy$ subject to $x+4y = 16$
b) $\max f(x,y) = x^2y$ subject to $2x^2+y^2 = 3$
c) $\max f(x,y,z) = xyz$ subject to $x^2+y^2 = 1$ and $x+z = 1$
d) $\min f(x,y) = x^2+y^2$ subject to $x^2+xy+y^2 = 3$
e) $\min f(x,y,z) = x^2+y^2+z^2$ subject to $3x+y+z = 5$ and $x+y+z = 1$
f) $\max/\min f(x,y,z) = x+y+z^2$ subject to $x^2+y^2+z^1 = 1$ and $y = 0$
g) $\max f(x,y,z) = xz+yz$ subject to $y^2+z^2 = 1$ and $xz = 3$
h) $\max x^2y^2z^2$ subject to $x^2+y^2+z^2 = 3$

5. Solve the Lagrange problems in Problem 4.

Solutions: Problem 1-2

a) $\boxed{\max\ xy \quad \text{s.t.}\ x+4y \le 16}$ (std. form)

$$\mathcal{L} = xy - \lambda(x+4y)$$

FOC	C	CSC
$\mathcal{L}'_x = y - \lambda = 0$	a) $x+4y=16$	$\lambda \ge 0$
$\mathcal{L}'_y = x - 4\lambda = 0$	b) $x+4y<16$	$\lambda = 0$

a) $\left.\begin{array}{l} x = 4\lambda \\ y = \lambda \end{array}\right\}$ $x+4y = 4\lambda + 4\lambda = 16$, $8\lambda = 16$, $\lambda = 2$ $\quad \boxed{\begin{array}{l} x=8 \\ y=2 \\ \lambda=2 \end{array}}$ $\lambda \ge 0$ ok

$f = xy = \underline{16}$

b) $\lambda = 0$ $\quad \boxed{\begin{array}{l} x=0 \\ y=0 \\ \lambda=0 \end{array}}$ $\quad x+4y = 0 < 16$ ok

$f = xy = \underline{0}$

Candidate for max: $\underline{x=8, y=2, \lambda=2}$ $\quad f=16$

Since $\boxed{\begin{array}{l} x=a \\ y=a \end{array}}$ is admissible when $a<0$ and

$f(a,a) = a^2 \to \infty$ when $a \to -\infty$, there is <u>no max</u>.

(If we try Method I, $\mathcal{L}(x,y,2)$ is <u>not concave</u>

— " — Method 2, $\{(x,y) : x+4y \le 16\}$ is <u>not bounded</u>

So these methods will not work.

b) $\boxed{\max\ x^2 y \quad \text{s.t.}\quad 2x^2+y^2 \leq 3}$ (std. form)

$\mathcal{L} = x^2y - \lambda(2x^2+y^2)$

$$\boxed{\begin{aligned} \mathcal{L}'_x &= 2xy - 4\lambda x = 0 \\ \mathcal{L}'_y &= x^2 - 2\lambda y = 0 \end{aligned}}$$

FOC

	C	CSC
a)	$2x^2+y^2=3$	$\lambda \geq 0$
b)	$2x^2+y^2<3$	$\lambda = 0$

a) $x^2-2\lambda y=0$

$\Downarrow$

$\cancel{y=0}$ or $\lambda = \frac{x^2}{2y}$

$y=0$: gives $x^2=0 \Rightarrow x=0$

$4\lambda x = 0$ ok

$y = \pm\sqrt{3/2} \Rightarrow$ impossible

$\Rightarrow y \neq 0$

$2xy = 4\cdot\frac{x^2}{2y}\cdot x$

$x\cdot 4y^2 = x\cdot 4x^2$

$x=0$ or $4x^2=4y^2$

$x = \pm y$

$x=0$ gives $2\lambda y = 0 \Rightarrow \lambda = 0$

$y^2=3 \Rightarrow y = \pm\sqrt{3}$

Solution: $(0,\pm\sqrt{3};0)$ $f=0$

$x\neq 0$: $x=\pm y$

$x^2=y^2 \Rightarrow 3x^2=3$

$x^2=1$

$x=\pm 1$

$y=\pm 1$

$\lambda = \frac{1}{2y} = \pm\frac{1}{2}$

Check: $\lambda \geq 0$ ok if $y=1$

Solutions: $(\pm1, 1; 1/2)$

$f=1$

b) $\lambda=0$: $2xy=0$

$x^2=0 \Rightarrow x=0$

$2x^2+y^2 = y^2 < 3 \Rightarrow -\sqrt{3}<y<\sqrt{3}$

Solution: $(0,y;0)$ $f=0$

(when $-\sqrt{3}<y<\sqrt{3}$)

Candidate for max: $(\pm1, 1; 1/2)$ $f=1$

$\{(x,y) : 2x^2+y^2 \leq 3\}$ is <u>bounded</u> since $2x^2 \leq 3 \Rightarrow x \in [-\sqrt{3/2}, \sqrt{3/2}]$

$y^2 \leq 3 \qquad y \in [-\sqrt{3}, \sqrt{3}]$

$\Rightarrow$ <u>there is a max</u> by E.V.T.

<u>NDCQ</u>: $2x^2+y^2 = 3$: $\text{rk}\begin{pmatrix} 4x & 2y \end{pmatrix} = 1$

$2x^2+y^2 < 3$: no condition

NDCQ fails : $2x^2+y^2=3$, $x=0, y=0$ (since $\text{rk}\begin{pmatrix} 4x & 2y \end{pmatrix} = 0$)

<u>impossible</u>

$\Downarrow$

* there is a max

* the max must occur either
  * a solution of KT-conditions $\leftarrow$ must be $(\pm 1, 1)$
  * a point where NDCQ fails $\leftarrow$ no points

<u>So $(\pm 1, 1)$ is max</u>

c) $\boxed{\max\ xyz \quad \text{s.t.} \quad \begin{matrix} x^2+y^2 \le 1 \\ x+z \ge 1 \end{matrix}}$

not std. form

rewrite $x+z \ge 1$ to $-x-z \le -1$

$$\mathcal{L} = xyz - \lambda_1(x^2+y^2) - \lambda_2(-x-z)$$

$$\boxed{\begin{aligned} \mathcal{L}'_x &= yz - 2\lambda_1 x + \lambda_2 = 0 \\ \mathcal{L}'_y &= xz - 2\lambda_1 y = 0 \\ \mathcal{L}'_z &= xy + \lambda_2 = 0 \end{aligned}}$$

FOC

	C	CSC
a)	$x^2+y^2=1,\ x+z=1$	$\lambda_1 \ge 0,\ \lambda_2 \ge 0$
b)	$x^2+y^2=1,\ x+z>1$	$\lambda_1 \ge 0,\ \lambda_2 = 0$
c)	$x^2+y^2<1,\ x+z=1$	$\lambda_1 = 0,\ \lambda_2 \ge 0$
d)	$x^2+y^2<1,\ x+z>1$	$\lambda_1 = 0,\ \lambda_2 = 0$

a) $\lambda_2 = -xy$

$\cancel{y=0}$ or $\lambda_1 = \frac{xz}{2y}$

$y=0$ gives: $\lambda_2 = 0$
$xz = 0$
$x = \pm 1 \Rightarrow z = 0 \Rightarrow x+z \neq 1$
impossible
$\Downarrow$
$y \neq 0$

$$yz - \frac{xz}{y}\cdot x + (-xy) = 0 \quad |\cdot y$$

$\left.\begin{matrix} z = 1-x \\ y^2 = 1-x^2 \end{matrix}\right\} \Rightarrow$ $y^2 z - x^2 z - xy^2 = 0$

$$(1-x^2)(1-x) - x^2(1-x) - x(1-x^2) = 0$$

$$(1-x)\cdot(1-x^2-x^2-x(1+x)) = 0$$

$\cancel{x=1}$ or $-3x^2 - x + 1 = 0$

$\Downarrow$
$y=0$
impossible

$$x = \frac{1 \pm \sqrt{13}}{-6} \simeq -0.77,\ 0.43$$

Compute $y, z, \lambda_1, \lambda_2$ for each value of x:

x	y	z	λ_1	λ_2
-0.77	± 0.64	1.77	∓ 0.49	∓ 0.87
0.43	± 0.90	0.57	± 0.14	± 0.22

$\lambda_1, \lambda_2 \ge 0$ gives two solns: $(-0.77, -0.64, 1.77;\ 0.49, 0.87)$ $f \simeq 0.87$

$(0.43, 0.90, 0.57;\ 0.14, 0.22)$ $f \simeq 0.22$

b) $\underline{\lambda_2 = 0}$: $xy = 0 \Rightarrow x = 0$ or $y = 0$

$\underline{x=0}$: $y = \pm 1, \lambda_1 = 0, z = 0 \Rightarrow (0, \pm 1, 0; 0, 0)$ $f = 0$

$x + z > 1$ not ok $\Rightarrow$ no sol'n.

$\underline{y=0}$: $x = \pm 1, z = 0, \lambda_1 = 0 \Rightarrow (\pm 1, 0, 0, 0, 0)$ $f = 0$

$x + z > 1$ not ok $\Rightarrow$ no sol'n

c) $\underline{\lambda_1 = 0}$: $\lambda z = 0 \Rightarrow x = 0$ or $z = 0$

$\underline{x=0}$: $z = 1, \lambda_2 = 0, y = 0 \Rightarrow \dfrac{(0, 0, 1; 0, 0)}{f = 0}$ $x^2 + y^2 < 1$ ok, $\lambda_2 \geq 0$ ok.

$\underline{z=0}$: $x = 1, \lambda_2 = 0, y = 0 \Rightarrow (1, 0, 0; 0, 0)$ $x^2 + y^2 < 1$ not ok. no sol'n

d) $\underline{\lambda_1 = \lambda_2 = 0}$: $xy = \lambda z = yz = 0 \Rightarrow x = 0, y = 0$ or $x = 0, z = 0$ or $y = 0, z = 0$

$\underline{x = 0, y = 0}$: $z \geq 1 \Rightarrow \underline{(0, 0, 1; 0, 0)\ f = 0}$ ok.

$\underline{x = 0, z = 0}$: $x + z > 1$ not ok $\Rightarrow$ no. sol'n.

$\underline{y = 0, z = 0}$: $x > 1 \Rightarrow x^2 + y^2 > 1 \Rightarrow$ no sol'n

Cand. for max: $x \cong -0.77$, $y \cong -0.64$, $z \cong 1.77$ $f \simeq 0.87$

If $x \simeq 0.77$, $y \simeq -0.64$, $z = a$ is admissible if $z = a \geq 1 - x \simeq 1.77$

$f(-0.77, -0.64, a) \simeq 0.49a \to \infty$ when $a \to \infty$

$\Downarrow$

there is no max

(Method 1 and Method 2 will not work)

d) $\boxed{\max\ xy \quad \text{s.t. } x^2+y^2 \le 1}$ (std. form)

$\mathcal{L} = xy - \lambda(x^2+y^2)$

$$\boxed{\begin{aligned} \mathcal{L}'_x &= y - 2\lambda x = 0 \\ \mathcal{L}'_y &= x - 2\lambda y = 0 \end{aligned}}$$

FOC

	C	CSC
i)	$x^2+y^2=1$	$\lambda \ge 0$
ii)	$x^2+y^2<1$	$\lambda = 0$

i) $y = 2\lambda x$

$x - 2\lambda\cdot(2\lambda x) = 0$

$x - 4\lambda^2 x = 0$ $\quad$ $\underline{x=0}$: $y=\pm 1$ and $y=0$ not possible

$\cancel{x=0}$ or $4\lambda^2 = 1$ $\quad \longleftarrow \quad \Rightarrow \underline{x \neq 0}$

$\lambda = \pm 1/2 \Rightarrow \lambda = 1/2$ (since $\lambda \ge 0$)

$x = y \Rightarrow x^2 = 1/2,\ y^2 = 1/2$

$x = \pm\sqrt{1/2},\ y = \pm\sqrt{1/2} \Rightarrow$ __Solutions:__ $(\sqrt{1/2}, \sqrt{1/2}; 1/2)$ $f = 1/2$

$(-\sqrt{1/2}, -\sqrt{1/2}; 1/2)$ $f = 1/2$

ii) $\lambda = 0$: $x = y = 0$

$x^2+y^2<1$ ok $\Rightarrow$ __Sol'n:__ $(0,0;0)$ $f=0$

__Cand. for max:__ $(\sqrt{1/2}, \sqrt{1/2}; 1/2)$ and $(-\sqrt{1/2}, -\sqrt{1/2}; 1/2)$

$\{(x,y): x^2+y^2 \le 1\}$ is __bounded__

NDCQ is satisfied: $\left\{\begin{array}{ll} x^2+y^2=1 & \text{rk}\,(2x\ \ 2y) = 1 \text{ holds; } (x,y) \neq (0,0) \\ x^2+y^2<1 & \text{no cond.} \end{array}\right.$

$\Downarrow$

$\underline{(\pm\sqrt{1/2}, \pm\sqrt{1/2}) \text{ is max}}$

(Method I will not work)

e) $\boxed{\max \; xyz \quad \text{s.t.} \quad \begin{array}{l} x+y+z \le 1 \\ x \ge 0 \\ y \ge 0 \\ z \ge 0 \end{array}}$ $\leftarrow$ not std. form $\left| \begin{array}{l} -x \le 0 \\ -y \le 0 \\ -z \le 0 \end{array} \right.$

$$\mathcal{L} = xyz - \lambda_1(x+y+z) + \lambda_2 x + \lambda_3 y + \lambda_4 z$$

$$\begin{array}{l} \mathcal{L}'_x = \\ \mathcal{L}'_y = \\ \mathcal{L}'_z = \end{array} \boxed{\begin{array}{l} yz - \lambda_1 + \lambda_2 = 0 \\ xz - \lambda_1 + \lambda_3 = 0 \\ xy - \lambda_1 + \lambda_4 = 0 \end{array}}$$

FOC

	C	CSC
i)	$x+y+z=1$	$\lambda_1 \ge 0$, $\lambda_2, \lambda_3, \lambda_4 \ge 0$; $\lambda_2 x = 0$, $\lambda_3 y = 0$, $\lambda_4 z = 0$
ii)	$x+y+z<1$	$\lambda_1 = 0$, $\lambda_2, \lambda_3, \lambda_4 \ge 0$; $\lambda_2 x = 0$, $\lambda_3 y = 0$, $\lambda_4 z = 0$

i) $x+y+z=1$:

Assume first $\lambda_1 > 0$. If $x=0$, then $\lambda_3 = \lambda_4 = \lambda_1 > 0$, so $y=z=0$.
But $x=0, y=0, z=0$ contradicts $x+y+z=1$. So no sol'n with $x=0$.
If $y=0$ or $z=0$, the same thing will happen.
Hence $x>0, y>0, z>0$ and $\lambda_2 = \lambda_3 = \lambda_4 = 0$
This gives $xy = xz = yz \Rightarrow x=y=z=1/3$

Sol'n: $(1/3, 1/3, 1/3; \lambda_1, 0, 0, 0)$ $(\lambda_1 > 0)$ $\quad f = 1/27$

Then assume $\lambda_1 = 0$: Then $yz = 0, \lambda_2 = 0$
$xz = 0, \lambda_3 = 0$
$xy = 0, \lambda_4 = 0$
(since each term in FOC is ≥ 0)

Sol'n: $(1,0,0; 0,0,0,0)$ $\quad f=0$
$(0,1,0; 0,0,0,0)$ $\quad f=0$
$(0,0,1; 0,0,0,0)$ $\quad f=0$

ii) $x+y+z<1$: $\lambda_1 = 0$ As above $yz=0, \lambda_2=0$
$xz=0, \lambda_3=0$
$xy=0, \lambda_4=0$

Solutions: $(x,0,0;0,0,0,0)$ $f=0$ $x<1$
$(0,y,0;0,0,0,0)$ $f=0$ $y<1$
$(0,0,z;0,0,0,0)$ $f=0$ $z<1$

Cand for max $f = 1/27$ at $(1/3, 1/3, 1/3; \lambda_1, 0, 0, 0)$ with $\lambda_1 > 0$

$\boxed{\begin{array}{l} x+y+z \le 1 \\ x \ge 0, y \ge 0, z \ge 0 \end{array}}$ defines a bounded set since $0 \le x \le 1$
$0 \le y \le 1$
$0 \le z \le 1$

NDCQ: $\mathrm{rk}\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = 3$

* At most three constraints binding simultaneously

$\Downarrow$

NDCQ holds for all pts.

(we must check that for all combinations of 3,2,1 binding constraints, the rank of the submatrix is 3,2,1 correspondingly)

$\Downarrow$

$(1/3, 1/3, 1/3)$ is max

f)

$$\boxed{\max\ xyz+z \quad \text{s.t.} \quad \begin{array}{l} x^2+y^2+z\le 6 \\ x\ge 0 \\ y\ge 0 \\ z\ge 0 \end{array}}$$

not std. form

$$\leftarrow \begin{cases} -x\le 0 \\ -y\le 0 \\ -z\le 0 \end{cases}$$

$$\mathcal{L} = xyz+z-\lambda(x^2+y^2+z)+v_1x+v_2y+v_3z$$

(write $\lambda=\lambda_1$, $v_1=\lambda_2$, $v_2=\lambda_3$, $v_3=\lambda_4$)

$$\begin{aligned} \mathcal{L}'_x &= \boxed{yz-2\lambda x+v_1=0} \\ \mathcal{L}'_y &= \boxed{xz-2\lambda y+v_2=0} \\ \mathcal{L}'_z &= \boxed{xy+1-\lambda+v_3=0} \end{aligned}$$

	C	CSC	
i)	$x^2+y^2+z=6$	$\lambda\ge 0$, $v_1,v_2,v_3\ge 0$	$v_1x=v_2y=v_3z=0$
ii)	$x^2+y^2+z<6$	$\lambda=0$, $v_1,v_2,v_3\ge 0$	$v_1x=v_2y=v_3z=0$

ii) $\underline{\lambda=0}$: $xy+1+v_3=0$

$xy\ge 0$, $v_3\ge 0 \Rightarrow$ not possible $\Rightarrow$ no sol'ns in case ii)

i) $\underline{x^2+y^2+z=6}$: If $\lambda=0$, then $xy+1+v_3=0$, a contradiction (as above) so $\lambda>0$.

$\underline{\text{Assume } x=0}$: $v_2=2\lambda y \Rightarrow \lambda=\dfrac{v_2}{2y}$ or $y=0$

$\underline{y=0}$: $z=6$, $v_1=0$, $v_2=0$, $v_3=0$, $\lambda=1 \Rightarrow \underline{(0,0,6;1,0,0,0)\quad f=6}$

$\underline{y\ne 0}$: $\lambda=v_2/2y$, $y>0 \Rightarrow v_2=0 \Rightarrow \lambda=0$, $v_3=-1$ contradiction ($v_3\ge 0$)

no sol'ns

$\underline{\text{Assume } x>0}$: $v_1=0$, $yz=2\lambda x>0$ (since $\lambda>0, x>0$)

$\Rightarrow y>0;\ z>0 \Rightarrow v_2=v_3=0$

$$\left.\begin{aligned} yz-2\lambda x&=0 \\ xz-2\lambda y&=0 \\ xy+1-\lambda&=0 \end{aligned}\right\} \qquad \lambda=xy+1$$

$$\begin{aligned} yz-2(xy+1)x&=0 \\ xz-2(xy+1)y&=0 \end{aligned}$$

$$(x+y)z-2(xy+1)(x+y)=0$$

$$(x+y)(z-2(xy+1))=0 \Rightarrow z-2(xy+1)=0$$

(since $x,y>0$)

$$\Rightarrow z=2(xy+1)=2\lambda$$

$yz - 2(xy+1)\lambda = 0$

$y \cdot 2\lambda - 2\lambda \cdot x = 0$

$(y-x) \cdot 2\lambda = 0 \Rightarrow y = x$ (since $\lambda > 0$)

$y = x$, $z = 2(xy+1) = 2(x^2+1) = 2x^2+2$

$\Rightarrow x^2+y^2+z = 6$

$x^2 + x^2 + 2(x^2+1) = 6$

$4x^2 + 2 = 6$

$4x^2 = 4 \Rightarrow x^2 = 1 \Rightarrow \underline{x=1, y=1, z=4}$

Sol'n: $\underline{(1,1,4; 2,0,0,0) \quad f = 8}$

Candidate for max: $\underline{(1,1,4; 2,0,0,0) \quad f=8}$

$x^2+y^2+z \le 6$, $x \ge 0$, $y \ge 0$, $z \ge 0$ defines a bounded set (since $x \in [0,\sqrt{6}]$, $y \in [0,\sqrt{6}]$, $z \in [0,6]$)

NDCQ:

$$\begin{pmatrix} 2x & 2y & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

* All four constraints cannot be binding
* Three binding constraints: rk = 3 in all cases

  Two binding constraints: rk = 2 —"—

  One binding —"— : rk = 1 —"—

$\Downarrow$

NDCQ holds for all admissible pts.

$\Downarrow$

$\underline{(1,1,4) \text{ is max}}$