Problem Session I Oct 02/Oct 09 2012

Themes:

Ⓐ **Matrix computations**

- compute the determinant and rank
- compute eigenvalues and eigenvectors
- diagonalize matrices, when it is possible
- determine definiteness of symmetric matrices
- Gaussian elimination and linear systems

Ⓑ **Quadratic forms**

- write in matrix form, and determine definiteness

Ⓒ **Vectors**

- linear independence

Matrices:

i) $A = \begin{pmatrix} 3 & -4 & -8 \\ -2 & 1 & 4 \\ 2 & -2 & -5 \end{pmatrix}$ ii) $A = \begin{pmatrix} 2 & -3 & 0 \\ -3 & 2 & 0 \\ 0 & 0 & 9 \end{pmatrix}$ iii) $A = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 2 \end{pmatrix}$

iv) $A = \begin{pmatrix} 1 & t & -2 \\ 2 & 4 & -t \\ -t & -4 & -4 \end{pmatrix}$ for $\begin{cases} \text{general } t \\ t = -2 \end{cases}$

v) $A = \begin{pmatrix} 1-a & 1 & -2a \\ 1 & 0 & 1 \\ 1 & a & -a \end{pmatrix}$ for $\begin{cases} \text{general } a \\ a = 1 \end{cases}$

vi) $A = \begin{pmatrix} t & t & 1 \\ 0 & t & 0 \\ 1 & 0 & t \end{pmatrix}$ for $\begin{cases} \text{general } t \\ t = 1 \end{cases}$

vii) $A = \begin{pmatrix} a-2 & 2(a-2) & -1 \\ 1 & a-1 & 2 \\ 2 & 2 & a+2 \end{pmatrix}$ for $\begin{cases} \text{general } a \\ a = 0 \end{cases}$

Problems

1) Compute $\det(A)$ and $rk(A)$ in i) – vii)

2) Find all eigenvalues and eigenvectors for A in i) – vii)

3) Determine if A is diagonalizable in i) – vii). If so, find a matrix P and D such that $P^{-1}AP = D$ is diagonal.

(Hint: In iv) – vii), the problems with parameters, it can be difficult to solve problem 2-3. Use the specific value of the parameter.)

4) Is $\begin{pmatrix} -2 \\ -2 \\ 0 \end{pmatrix}$ an eigenvector for $\begin{pmatrix} 3 & -4 & 8 \\ -2 & 1 & 4 \\ 2 & -2 & +5 \end{pmatrix}$. If so, what is the eigenvalue?

" $\begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$ — " — $\begin{pmatrix} 1 & t & -2 \\ 2 & 4 & -t \\ -t & -4 & -4 \end{pmatrix}$ for any values of t? If so, what is the eigenvalue?

5) Are the vectors linearly independent?

$$v_1 = \begin{pmatrix} 1 \\ 3 \\ -1 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} -2 \\ 2 \\ 0 \\ 1 \end{pmatrix} \quad v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \quad v_4 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ -1 \end{pmatrix}$$

6) Assume that $\underline{a}, \underline{b}, \underline{c}$ are linearly independent vectors. Show that

$$3\underline{a} - 2\underline{b} + \underline{c}, \quad 2\underline{a} + \underline{b}, \quad \underline{a} + 2\underline{b}$$

are also linearly independent.

7) For what values of a are the vectors linearly independent?

$$\underline{v}_1 = \begin{pmatrix} +1 \\ a \\ 1 \\ 0 \end{pmatrix} \quad \underline{v}_2 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix} \quad \underline{v}_3 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ a \end{pmatrix} \quad \underline{v}_4 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ -1 \end{pmatrix}$$

8) Let $Q(x_1, x_2, x_3) = ax_1^2 + 4x_1x_2 + x_2^2 + 2x_1x_3 + 3x_3^2$. For what values of a is Q positive definite?

Solution:

i)
$$\begin{vmatrix} 3 & -4 & -8 \\ -2 & 1 & 4 \\ 2 & -2 & -5 \end{vmatrix} = 3(-5+8)+2(20-16)+2(-16+8)$$
$$= 9+8-16 = \underline{1}$$

$|A| \neq 0 \Rightarrow \text{rk } \underline{A=3}$

ii)
$$\begin{vmatrix} 2 & -3 & 0 \\ -3 & 2 & 0 \\ 0 & 0 & 9 \end{vmatrix} = 9\cdot(4-9) = \underline{-45}$$

$|A| \neq 0 \Rightarrow \text{rk } A = \underline{3}$

iii)
$$\begin{vmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 2 \end{vmatrix} = 4\cdot(6-4)+2(-4) = 8-8 = \underline{0}$$

$|A| = 0 \Rightarrow \text{rk } A < 3$

$\begin{vmatrix} 4 & -2 \\ -2 & 3 \end{vmatrix} = 12-4 = 8 \neq 0 \Rightarrow \text{rk } A = \underline{2}$

iv)
$$\begin{vmatrix} 1 & t & -2 \\ 2 & 4 & -t \\ -t & -4 & -4 \end{vmatrix} = 1\cdot(-16-4t)-2(-4t-8)-t(-t^2+8)$$
$$= \cancel{-16}-4t+\cancel{8t}+\cancel{16}+t^3-\cancel{8t}$$
$$= \underline{t^3-4t = t(t^2-4) = t(t-2)(t+2)}$$

$|A| = 0 \iff t = 0, -2, 2$

$\underline{rk\,A = 3}$ when $t \neq 0, -2, 2$

$\underline{t=0}$: $A = \begin{pmatrix} 1 & 0 & -2 \\ 2 & 4 & 0 \\ 0 & -4 & -4 \end{pmatrix}$ $\begin{vmatrix} 1 & 0 \\ 2 & 4 \end{vmatrix} = 4 \neq 0$

$\Rightarrow \underline{rk\,A = 2}$ for $t=0$

$\underline{t=-2}$: $A = \begin{pmatrix} 1 & -2 & -2 \\ 2 & 4 & 2 \\ 2 & -4 & -4 \end{pmatrix}$ $\begin{vmatrix} 1 & -2 \\ 2 & 4 \end{vmatrix} = 4+4 = 8 \neq 0$

$\Rightarrow \underline{rk\,A = 2}$ for $t=-2$

$\underline{t=2}$: $A = \begin{pmatrix} 1 & 2 & -2 \\ 2 & 4 & -2 \\ -2 & -4 & -4 \end{pmatrix}$ $\begin{vmatrix} 2 & -2 \\ 4 & -2 \end{vmatrix} = -4+8 = 4 \neq 0$

$\Rightarrow \underline{rk\,A = 2}$ for $t=2$

$\underline{\text{Concl:}}$

$$rk\,A = \begin{cases} 3, & t \neq 0, -2, 2 \\ 2, & t = 0, -2, 2 \end{cases}$$

v)

$$\begin{vmatrix} 1-a & 1 & -2a \\ 1 & 0 & 1 \\ 1 & a & -a \end{vmatrix} = -1(-a-1) - a((1-a)+2a)$$

$$= a+1 - a(1+a) = \underline{1-a^2}$$

$$= \underline{1-a^2} = \underline{(1-a)(1+a)}$$

$|A| = 0 \iff a = -1, +1$

$rk\,A = 3$ for $a \neq \pm 1$

$\underline{a=-1}$: $A = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}$ $\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$

$\Rightarrow rk\,A = 2$ for $a=-1$

$\underline{a=1}$: $A = \begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}$ $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0 \Rightarrow rk\,A = 2$ for $a=1$

Concl: $\operatorname{rk} A = \begin{cases} 3 & , a \neq -1, +1 \\ 2 & , a = -1, +1 \end{cases}$

vi) $$|A| = \begin{vmatrix} t & t & 1 \\ 0 & t & 0 \\ 1 & 0 & t \end{vmatrix} = t\cdot(t^2-1) = t(t+1)(t-1)$$

$|A| = 0 \iff t=0, t=1, t=-1$

$\operatorname{rk} A = 3$ if $t \neq 0, 1, -1$

t=0: $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0 \Rightarrow \operatorname{rk} A = 2$ for $t=0$

t=1: $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ $\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \neq 0 \Rightarrow \operatorname{rk} A = 2$ for $t=1$

t=-1: $A = \begin{pmatrix} -1 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$ $\begin{vmatrix} -1 & -1 \\ 0 & -1 \end{vmatrix} = 1 \neq 0 \Rightarrow \operatorname{rk} A = 2$ for $t=-1$

Concl: $\operatorname{rk} A = \begin{cases} 3 & , t \neq 0, 1, -1 \\ 2 & , t = 0, 1, -1 \end{cases}$

vii) $$\begin{vmatrix} a-2 & 2(a-2) & -1 \\ 1 & a-1 & 2 \\ 2 & 2 & a+2 \end{vmatrix} = (a-2)\big((a-1)(a+2)-4\big)$$

$$-1\big(2(a-2)(a+2)+2\big) + 2\big(4(a-2)+(a-1)\big)$$

$$= (a-2)(a^2+a-6) - (a-2)(2a+4) - 2 + (a-2)\cdot 8 + 2a - 2$$

$$= (a-2)(a^2+a-6-2a-4+8) + 2a-4$$

$$= (a-2)(a^2+a-6-2a-4+8+2)$$

$= (a-2)(a^2-a) = \underline{(a-2)\,a\,(a-1)}$

$|A| = 0 \iff a = 2, 0, 1$

$\text{rk}\, A = 3$ for $a \neq 2, 0, 1$

$\underline{a=2}$: $A = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{pmatrix}$ $\begin{vmatrix} 0 & -1 \\ 1 & 2 \end{vmatrix} = +1 \neq 0 \Rightarrow \underline{\text{rk}\, A = 2}$ for $a=2$

$\underline{a=0}$: $A = \begin{pmatrix} -2 & -4 & -1 \\ 1 & -1 & 2 \\ 2 & 2 & 2 \end{pmatrix}$ $\begin{vmatrix} -2 & -4 \\ 1 & -1 \end{vmatrix} = 6 \neq 0 \Rightarrow \underline{\text{rk}\, A = 2}$ for $a=0$

$\underline{a=1}$: $A = \begin{pmatrix} -1 & -2 & -1 \\ 1 & 0 & 2 \\ 2 & 2 & 2 \end{pmatrix}$ $\begin{vmatrix} -1 & -2 \\ 1 & 0 \end{vmatrix} = 2 \neq 0 \Rightarrow \text{rk}\, A = 2$ for $a=1$

$\underline{\text{Concl}}$: $\text{rk}\, A = \begin{cases} 3, & a \neq 2, 1, 0 \\ 2, & a = 2, 1, 0 \end{cases}$

Solución:

i) $\begin{vmatrix} 3-\lambda & -4 & -8 \\ -2 & 1-\lambda & 4 \\ 2 & -2 & -5-\lambda \end{vmatrix} = 0$

$(3-\lambda)\cdot\Big((1-\lambda)(-5-\lambda) + 8 \Big) + 2\Big(-4(-5-\lambda) - 16\Big) + 2\Big(-16 + 8 - 8\lambda\Big)$

$= (3-\lambda)(1-\lambda)(-5-\lambda) + \Big(24 - \cancel{8\lambda} + 8 + \cancel{8\lambda} - 16 - 16\lambda \Big)$

$= (3-\lambda)(1-\lambda)(-5-\lambda) + (16 - 16\lambda)$

$= (3-\lambda)(1-\lambda)(-5-\lambda) + 16(1-\lambda)$

$= (1-\lambda)\cdot\Big((3-\lambda)(-5-\lambda) + 16 \Big) = (1-\lambda)\cdot(\lambda^2 + 2\lambda + 1)$

$= (1-\lambda)\cdot(\lambda+1)^2 = 0$

$-\lambda^3 - \lambda^2 + \ldots + 1 = 0$

Eigenvalues: $\underline{\underline{\lambda = 1}}$, $\underline{\underline{\lambda = -1}}$

$(\lambda_1 = 1 \quad \lambda_2 = -1, \lambda_3 = -1)$

Eigenvectors:

$\lambda = 1$: $\begin{pmatrix} 2 & -4 & -8 \\ -2 & 0 & 4 \\ 2 & -2 & -6 \end{pmatrix} \to \begin{pmatrix} 2 & -4 & -8 \\ 0 & -4 & -4 \\ 0 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 2 & 0 & -4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$$\left.\begin{array}{lll} 2x - 4z = 0 & & x = 2z \\ y + z = 0 & & y = -z \\ z \text{ free} & & z = z \end{array}\right\} \quad \underline{x} = z\cdot\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2z \\ -z \\ z \end{pmatrix}$

$\underline{\lambda=-1}$: $\begin{pmatrix} 4 & -4 & -8 \\ -2 & 2 & 4 \\ 2 & -2 & -4 \end{pmatrix} \to \begin{pmatrix} 4 & -4 & -8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$\left.\begin{array}{r} x - y - 2z = 0 \\ y \text{ free} \\ z \text{ free} \end{array}\right\} \quad \begin{array}{l} x = y+2z \\ y = y \\ z = z \end{array} \qquad \underline{x} = \begin{pmatrix} y+2z \\ y \\ z \end{pmatrix}$$

$$= y \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

ii)

$$\begin{vmatrix} 2-\lambda & -3 & 0 \\ -3 & 2-\lambda & 0 \\ 0 & 0 & 9-\lambda \end{vmatrix} = (9-\lambda)\cdot\left((2-\lambda)(2-\lambda) - 9 \right)$$

$$= (9-\lambda)(\lambda^2 - 4\lambda - 5) = (9-\lambda)(\lambda - 5)(\lambda+1) = 0$$

Eigenvalues: $\underline{\underline{\lambda = 9,\ \lambda = 5,\ \lambda = -1}}$

Eigenvectors:

$\underline{\lambda = 9}$: $\begin{pmatrix} -7 & -3 & 0 \\ -3 & -7 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} -7 & -3 & 0 \\ 0 & * & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x = y = 0$, $z = $ free

$* \neq 0$

$$\underline{x} = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} = z \cdot \underline{\underline{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}}$$

$\underline{\lambda = 5}$: $\begin{pmatrix} -3 & -3 & 0 \\ -3 & -3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad \begin{array}{l} x+y=0 \\ z=0 \end{array} \quad \begin{array}{l} x = -y \\ y = y \text{ (free} \\ z = 0 \end{array}$

$$\underline{x} = \begin{pmatrix} -y \\ y \\ 0 \end{pmatrix} = y \cdot \underline{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}$$

$\underline{\lambda=-1}$: $\begin{pmatrix} 3 & -3 & 0 \\ -3 & 3 & 0 \\ 0 & 0 & 10 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x-y=0$, $z=0$ (y free)

$$\underline{x} = \begin{pmatrix} y \\ y \\ 0 \end{pmatrix} = y \cdot \underline{\underline{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}}$$

iii) $\begin{vmatrix} 4-\lambda & -2 & 0 \\ -2 & +3-\lambda & -2 \\ 0 & -2 & 2-\lambda \end{vmatrix} = (4-\lambda)\Big((3-\lambda)(2-\lambda)-4\Big)+2\Big(-2(2-\lambda)\Big)$

$= (4-\lambda)\cdot(3-\lambda)(2-\lambda) + (4-\lambda)(-4) - 4(2-\lambda)$

$= (4-\lambda)(3-\lambda)(2-\lambda) + 4\lambda - 16 - 8 + 4\lambda$

$= (4-\lambda)(3-\lambda)(2-\lambda) + 8\lambda - 24 = (4-\lambda)(3-\lambda)(2-\lambda) + 8(\lambda-3)$

$= (\lambda-3)\cdot\Big(-(4-\lambda)(2-\lambda)+8\Big) = (\lambda-3)\cdot(-\lambda^2+6\lambda) = \underline{(\lambda-3)(-\lambda)(\lambda-6)=0}$

Eigenvalues: $\underline{\lambda_1=3}$ $\quad \underline{\lambda_2=0}$ $\quad \underline{\lambda_3=6}$

Eigenvectors:

$\underline{\lambda=3}$: $\begin{vmatrix} 1 & -2 & 0 \\ -2 & 0 & -2 \\ 0 & -2 & -1 \end{vmatrix} \to \begin{pmatrix} 1 & -2 & 0 \\ 0 & -4 & -2 \\ 0 & -2 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 0 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x-2y=0$, $-2y-z=0$ (z free)

$$\left.\begin{aligned} x &= +2y = +2\left(-\tfrac{1}{2}z\right) = -z \\ y &= -\tfrac{1}{2}z \\ z &\ \text{free} \end{aligned}\right\} \quad \underline{x} = \begin{pmatrix} -z \\ -1/2\, z \\ z \end{pmatrix} = z \cdot \underline{\underline{\begin{pmatrix} -1 \\ -1/2 \\ 1 \end{pmatrix}}}$$

$\lambda=0$: $\begin{pmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 2 \end{pmatrix} \to \begin{pmatrix} -2 & 3 & -2 \\ 0 & 4 & -4 \\ 0 & -2 & 2 \end{pmatrix} \to \begin{pmatrix} -2 & 3 & -2 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad$ $-2x+3y-2z=0$, $-2y+2z=0$ (z free)

$$\left.\begin{aligned} x &= \frac{3y-2z}{2} = \frac{1}{2}z \\ y &= z \\ z &= z \end{aligned}\right\} \quad \underline{x} = \begin{pmatrix} \frac{1}{2}z \\ z \\ z \end{pmatrix} = z \cdot \underline{\underline{\begin{pmatrix} \frac{1}{2} \\ 1 \\ 1 \end{pmatrix}}}$$

$\lambda=6$: $\begin{pmatrix} -2 & -2 & 0 \\ -2 & -3 & -2 \\ 0 & -2 & -4 \end{pmatrix} \to \begin{pmatrix} -2 & -2 & 0 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \end{pmatrix} \to \begin{pmatrix} -2 & -2 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad$ $-2x-2y=0$, $-y-2z=0$ (z free)

$$\left.\begin{aligned} x &= -y = 2z \\ y &= -2z \\ z &= z \end{aligned}\right\} \quad \underline{x} = \begin{pmatrix} 2z \\ -2z \\ z \end{pmatrix} = z \cdot \underline{\underline{\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}}}$$

iv) With $t=-2$:

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & t & -2 \\ 2 & 4-\lambda & -t \\ -t & -4 & -4-\lambda \end{vmatrix} \overset{t=-2}{=} \begin{vmatrix} 1-\lambda & -2 & -2 \\ 2 & 4-\lambda & 2 \\ 2 & -4 & -4-\lambda \end{vmatrix}$$

$$= (1-\lambda)\cdot\left((4-\lambda)(-4-\lambda)+8\right) - 2\left(-2(-4-\lambda)-8\right) + 2\left(-4+2(4-\lambda)\right)$$

$$= (1-\lambda)(4-\lambda)(-4-\lambda) + 8(1-\lambda) - 2(\cancel{8}+2\lambda-\cancel{8}) + 2(-4+8-2\lambda)$$

$$= (1-\lambda)(4-\lambda)(-4-\lambda) + (8+8-8\lambda-4\lambda-4\lambda) = (1-\lambda)(4-\lambda)(-4-\lambda)+16-16\lambda$$

$$= (1-\lambda)(4-\lambda)(-4-\lambda)+16(1-\lambda) = (1-\lambda)\left((4-\lambda)(-4-\lambda)+16\right) = (1-\lambda)(-16+\lambda^2+16)$$

$$= \lambda^2(1-\lambda) = 0$$

Eigenvalues: $\underline{\lambda=0}$ (mult 2), $\underline{\lambda=1}$

Eigenvectors:

$\lambda=0$: $\begin{pmatrix} 1 & -2 & -2 \\ 2 & 4 & 2 \\ 2 & -4 & -4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & -2 \\ 0 & 8 & 6 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 4 & 3 \\ 0 & 0 & 0 \end{pmatrix}$

$x = \frac{1}{2}z$

(z free)

$4y+3z=0$

$\Downarrow$

$y = -\frac{3}{4}z$

$\left.\begin{array}{l} x = \frac{1}{2}z \\ y = -\frac{3}{4}z \\ z = z \end{array}\right\}$ $\underline{x} = \begin{pmatrix} \frac{1}{2}z \\ -\frac{3}{4}z \\ z \end{pmatrix} = z \cdot \underline{\underline{\begin{pmatrix} 1/2 \\ -3/4 \\ 1 \end{pmatrix}}}$

$\lambda=1$: $\begin{pmatrix} 0 & -2 & -2 \\ 2 & 3 & 2 \\ 2 & -4 & -5 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & -7 & -7 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$2x - z = 0$

$y + z = 0$

(z free)

$\left.\begin{array}{l} x = \frac{1}{2}z \\ y = -z \\ z = z \end{array}\right\}$ $\underline{x} = \begin{pmatrix} \frac{1}{2}z \\ -z \\ z \end{pmatrix} = z \cdot \underline{\underline{\begin{pmatrix} 1/2 \\ -1 \\ 1 \end{pmatrix}}}$

V) With a=1:

$$|A-\lambda I| = \begin{vmatrix} 1-a-\lambda & 1 & -2a \\ 1 & -\lambda & 1 \\ 1 & a & -a-\lambda \end{vmatrix} \overset{a=1}{=} \begin{vmatrix} -\lambda & 1 & -2 \\ 1 & -\lambda & 1 \\ 1 & 1 & -1-\lambda \end{vmatrix}$$

$$= -\lambda(-\lambda(-1-\lambda)-1) - 1(-1-\lambda+2) + 1(1-2\lambda)$$

$$= \lambda^2(-1-\lambda) + \cancel{\lambda} \cancel{-1} + \cancel{\lambda} \cancel{+1} \cancel{-2\lambda} = \lambda^2(-1-\lambda) = 0$$

Eigenvalue: $\underline{\lambda=0}$ $\lambda = \underline{-1}$

(mult. 2)

Eigenvectors:

$\lambda=0$: $\begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x+z=0$, $y-2z=0$ (z free)

$$\left.\begin{aligned} x &= -z \\ y &= 2z \\ z &= z \end{aligned}\right\} \quad \underline{x} = \begin{pmatrix} -z \\ 2z \\ z \end{pmatrix} = z \cdot \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

$\lambda=-1$: $\begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x+y=0$, $z=0$ (y free)

$$\left.\begin{aligned} x &= -y \\ y &= y \\ z &= 0 \end{aligned}\right\} \quad \underline{x} = \begin{pmatrix} -y \\ y \\ 0 \end{pmatrix} = y \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

vi) With $t=1$:

$$|A-\lambda I| = \begin{vmatrix} t-\lambda & t & 1 \\ 0 & t-\lambda & 0 \\ 1 & 0 & t-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\cdot\left((1-\lambda)^2-1\right)$$

$$= (1-\lambda)(\lambda^2-2\lambda) = (1-\lambda)\lambda(\lambda-2) = 0$$

Eigenvalues: $\lambda=1$, $\lambda=0$, $\lambda=2$

Eigenvectors:

$\lambda=1$: $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x=0$, $y=-z$ (z free) $\quad \underline{x} = \begin{pmatrix} 0 \\ -z \\ z \end{pmatrix} = z\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

$\underline{\lambda=0}$: $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x=-z$, $y=0$, z free $\quad \underline{x} = \begin{pmatrix} -z \\ 0 \\ z \end{pmatrix} = z \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\underline{\lambda=2}$: $\begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad \cancel{y} x=z$, $y=0$, z free $\quad \underline{x} = \begin{pmatrix} z \\ 0 \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

(Ex. vi) is also possible to solve for general t)

vii) With $a=0$:

$$|A-\lambda I| = \begin{vmatrix} a-2-\lambda & 2(a-2) & -1 \\ 1 & a-1-\lambda & 2 \\ 2 & 2 & a+2-\lambda \end{vmatrix} \overset{a=0}{=} \begin{vmatrix} -2-\lambda & -4 & -1 \\ 1 & -1-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{vmatrix}$$

$$= (-2-\lambda)\left((-1-\lambda)(2-\lambda)-4\right) - 1\left(-4(2-\lambda)+2\right) + 2\left(-8+(-1-\lambda)\right)$$

$$= (-2-\lambda)(-1-\lambda)(2-\lambda) + (-4)(-2-\lambda) - 1(-6+4\lambda) + 2(-9-\lambda)$$

$$= (-2-\lambda)(-1-\lambda)(2-\lambda) + (-2\lambda-4) = (-2-\lambda)(-1-\lambda)(2-\lambda) + 2\cdot(-2-\lambda)$$

$$= (-2-\lambda)\cdot\left((-1-\lambda)(2-\lambda)+2\right) = -(\lambda+2)(\lambda^2-\lambda) = -(\lambda+2)\lambda(\lambda-1) = 0$$

Eigenvalues: $\underline{\lambda=-2}$, $\underline{\lambda=0}$, $\underline{\lambda=1}$

Eigenvectors:

$\underline{\lambda=-2}$: $\begin{pmatrix} 0 & -4 & -1 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 \\ 0 & -4 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x = -y-2z = +\frac{1}{4}z - 2z = -\frac{7}{4}z$, $y = -\frac{1}{4}z$, (z free)

$$\underline{x} = \begin{pmatrix} -7/4\, z \\ -1/4\, z \\ z \end{pmatrix} = z \cdot \begin{pmatrix} -7/4 \\ -1/4 \\ 1 \end{pmatrix}$$

$\underline{\lambda = 0}$: $\begin{pmatrix} -2 & -4 & -1 \\ 1 & -1 & 2 \\ 2 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 2 \\ 0 & -6 & 3 \\ 0 & 4 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

$x - y + 2z = 0$
$2y - z = 0$
(z free)

$x = y - 2z = \frac{1}{2}z - 2z = -\frac{3}{2}z$
$y = \frac{1}{2}z$
$z = z$

$$\underline{x} = \begin{pmatrix} -3/2\, z \\ 1/2\, z \\ z \end{pmatrix} = z \underline{\underline{\begin{pmatrix} -3/2 \\ 1/2 \\ 1 \end{pmatrix}}}$$

$\underline{\lambda = 1}$: $\begin{pmatrix} -3 & -4 & -1 \\ 1 & -2 & 2 \\ 2 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 2 \\ 0 & -10 & 5 \\ 0 & 6 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

$x = 2y - 2z = z - 2z = -z$
$y = \frac{1}{2}z$
(z free)

$$\underline{x} = \begin{pmatrix} -z \\ \frac{1}{2}z \\ z \end{pmatrix} = z \cdot \underline{\underline{\begin{pmatrix} -1 \\ 1/2 \\ 1 \end{pmatrix}}}$$

<u>Hints</u>: Problems with eigenvalues / eigenvectors

a) $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = \det(A)$ – use this to check your calculations

b) If $\det(A) = 0$, then $\lambda = 0$ is eigenvalue since $|A| = 0 \iff \lambda_1 \lambda_2 \lambda_3 = 0 \iff \lambda_1 = 0$ or $\lambda_2 = 0$ or $\lambda_3 = 0$

c) If λ is eigenvalue, then the corresponding system $(A - \lambda I) \cdot \underline{x} = \underline{0}$ has <u>at least</u> one degree of freedom.

c) Diagonalizable matrices

A diagonalizable $\Leftrightarrow$ there are $\underline{n}$ lin. independent eigenvectors for A
(A $n \times n$ matrix)

Check: Find all eigenvalues and all eigenvectors

A diagonalizable $\Leftrightarrow$

$$\begin{aligned} & n_1 = \# \text{ degrees of freedom for } \lambda_1 \\ + & n_2 = \text{—} \text{ " } \text{—} \quad \lambda_2 \\ + & n_3 = \text{—} \text{ " } \text{—} \quad \lambda_3 \\ \hline & n = 3 \end{aligned}$$

If A is symmetric, it is always diagonalizable
If A has n distinct eigenvectors, it is always diagonalizable.

Exc: Determine if A is diagonalizable in i) – vii).
(Use specific numerical value for parameters)

i) ~~$\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = -1$~~ $\underline{\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = -1}$ ($\lambda = -1$ mult. 2)

Count degrees of freedom:

$$\underset{\substack{\uparrow \\ \text{for } \lambda = 1}}{1} + \underset{\substack{\uparrow \\ \lambda = -1}}{2} = 3 \Rightarrow A \text{ \underline{diagonalizable}}$$

ii) A symmetric $\Rightarrow$ A <u>diagonalizable</u> (Also $1+2=3$)

iii) A has three distinct eigenvalues $\Rightarrow$ A <u>diagonalizable</u> $\left\{ \begin{array}{l} \text{(Also, A symmetric)} \\ \text{(Also, } 1+1+1=3) \end{array} \right.$

iv) <u>with $t=-2$:</u> A <u>not</u> diagonalizable ($1+1=2 \neq 3$)

v) With $a=1$: A not diagonalizable ($1+1=2 \neq 3$)

vi) With $t=1$: A diagonalizable

since three distinct eigenvalues (Also, $1+1+1=3$)

vii) With $a=0$: A diagonalizable

since three distinct eigenvalues (Also $1+1+1=3$)

Solution:

i) $A\underline{v} = \begin{pmatrix} 3 & -4 & 8 \\ -2 & 1 & 4 \\ 2 & -2 & -5 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$

$\lambda \underline{v} = \lambda \cdot \begin{pmatrix} -2 \\ -2 \\ 0 \end{pmatrix}$

Equality hold is $\underline{\lambda = -1}$.

Yes, $\underline{v}$ is eigenvecter with eigenvalue $\underline{\lambda = -1}$

ii) $A\underline{v} = \begin{pmatrix} 1 & t & -2 \\ 2 & 4 & -t \\ -t & -4 & -4 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 + 2t + 4 \\ -2 + 8 + 2t \\ t - 8 + 8 \end{pmatrix} = \begin{pmatrix} 2t + 3 \\ 2t + 6 \\ t \end{pmatrix}$

$\lambda \underline{v} = \lambda \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -\lambda \\ 2\lambda \\ -2\lambda \end{pmatrix}$

Equality holds if $\begin{cases} 2t + 3 = -\lambda \\ 2t + 6 = 2\lambda \\ t = -2\lambda \end{cases} \Rightarrow t = -2\lambda$

$\Rightarrow 2(-2\lambda) + 3 = -\lambda$

$-4\lambda + 3 = -\lambda$

$3 = -3\lambda$

$\underline{\lambda = +1}$

$\underline{t = -2}$

Yes, $\underline{v}$ is eigenvecter for $t = -2$, and the eigenvalue is $\lambda = 1$. ($\underline{v}$ is not eigenvecter for $t \neq -2$).

Solutions:

1. $av_1 + bv_2 + cv_3 + dv_4 = 0$

$$\begin{pmatrix} 1 & -2 & -1 & 1 \\ -1 & 0 & 1 & 2 \\ 3 & 2 & 0 & -1 \\ 0 & 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Let

$$A = \begin{pmatrix} 1 & -2 & -1 & 1 \\ -1 & 0 & 1 & 2 \\ 3 & 2 & 0 & -1 \\ 0 & 1 & 1 & -1 \end{pmatrix} \rightarrow detA = \begin{vmatrix} 1 & -2 & -1 & 1 \\ -1 & 0 & 1 & 2 \\ 3 & 2 & 0 & -1 \\ 0 & 1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & -1 & 1 \\ 0 & -2 & 0 & 3 \\ 3 & 2 & 0 & -1 \\ 1 & -1 & 0 & 0 \end{vmatrix} =$$

$$-\begin{vmatrix} 0 & -2 & 3 \\ 3 & 2 & -1 \\ 1 & -1 & 0 \end{vmatrix} = 13 \neq 0$$ $\rightarrow$ vectors are linearly independent

2.

$x(3\mathbf{a}-2\mathbf{b}+\mathbf{c})+\mathrm{y}(2\mathbf{a}+\mathbf{b})+\mathrm{z}(\mathbf{a}+2\mathbf{b}\,)=0$

$$\rightarrow (3\mathrm{x}+2\mathrm{y}+\mathrm{z})\mathbf{a}+(-2\mathrm{x}+\mathrm{y}+2\mathrm{z})\mathbf{b}+\mathrm{x}\mathbf{c}=0$$

As vectors a, b, c need to be linearly independent, we have:

$$\begin{cases} 3\mathrm{x}+2\mathrm{y}+\mathrm{z}=0 \\ -2\mathrm{x}+\mathrm{y}+2\mathrm{z}=0 \\ x=0 \end{cases} \rightarrow \mathrm{x}=\mathrm{y}=\mathrm{z}=0 \rightarrow \text{the given vectors are linearly independent}$$

3.

$$\begin{vmatrix} -1 & 1 & 0 & 1 \\ a & 2 & -1 & 1 \\ 1 & -1 & 0 & 2 \\ 0 & 0 & a & -1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 1 \\ a+1 & 1 & -1 & 1 \\ 3 & -3 & 0 & 2 \\ -1 & 1 & a & -1 \end{vmatrix} = -\begin{vmatrix} a+1 & 1 & -1 \\ 3 & -3 & 0 \\ -1 & 1 & a \end{vmatrix}$$

$$= 3a(a+2)$$

$$detA = 0 \rightarrow a \in \{-2,0\}$$

4.

$$A = \begin{pmatrix} a & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$

The following conditions must be satisfied:

$$\Delta_1 = a > 0, \Delta_2 = \begin{vmatrix} a & 2 \\ 2 & 1 \end{vmatrix} = a - 4 > 0, \Delta_3 = 3a - 13 > 0$$

This gives $a > \frac{13}{3}$