Problem Sheet 7 with Solutions
GRA 6035 Mathematics

BI Norwegian Business School

Problems

1. Find all extremal points for the function $f(x,y,z) = x^4 + y^4 + z^4 + x^2 + y^2 + z^2$.

2. Show that the function $f(x,y) = x^3 + y^3 - 3x - 2y$ defined on the convex set $S = \{(x,y) : x > 0,\ y > 0\}$ is (strictly) convex, and find its global minimum.

3. A company produces two output goods, denoted by A and B. The cost per day is

$$C(x,y) = 0.04x^2 - 0.01xy + 0.01y^2 + 4x + 2y + 500$$

when x units of A and y units of B are produced ($x > 0,\ y > 0$). The firm sells all it produces at prices 13 per unit of A and 8 per unit of B. Find the profit function π and the values of x and y that maximizes profit.

4. The function $f(x,y,z) = x^2 + 2xy + y^2 + z^3$ is defined on $S = \{(x,y,z) : z > 0\}$. Show that S is a convex set. Find the stationary points of f and the Hessian matrix. Is f convex or concave? Does f have a global extremal point?

5. Show that the function $f(x,y,z) = x^4 + y^4 + z^4 + x^2 - xy + y^2 + yz + z^2$ is convex.

6. Find all local extremal points for the function $f(x,y,z) = -2x^4 + 2yz - y^2 + 8x$ and classify their type.

7. The function $f(x,y,z) = x^2 + y^2 + 3z^2 - xy + 2xz + yz$ defined on $\mathbb{R}^3$ has only one stationary point. Show that it is a local minimum.

8. Find all local extremal points for the function $f(x,y) = x^3 - 3xy + y^3$ and classify their type.

9. The function $f(x,y,z) = x^3 + 3xy + 3xz + y^3 + 3yz + z^3$. Find all local extremal points for f and classify their type.

10. Find the solution $(x^*(a), y^*(a), z^*(a))$ to the Lagrange problem

$$\max f(x,y,z) = 100 - x^2 - y^2 - z^2 \text{ subject to } x + 2y + z = a$$

and let $\lambda(a)$ be the corresponding Lagrange multiplier. Show that

$$\lambda(a) = \frac{\partial f^*(a)}{\partial a}$$

where $f^*(a) = f(x^*(a), y^*(a), z^*(a), \lambda(a))$ is the optimal value function.

11. Solve the Lagrange problem

$$\max f(x,y,z) = x + 4y + z \text{ subject to } \begin{cases} x^2 + y^2 + z^2 = 216 \\ x + 2y + 3z = 0 \end{cases}$$

Use the Lagrange multiplier to estimate the new maximum value when the constraints are changed to $x^2 + y^2 + z^2 = 215$ and $x + 2y + 3z = 0.1$.

12. Final Exam in GRA6035 10/12/2010, Problem 1
We consider the function $f(x,y,z) = x^2e^x + yz - z^3$.

a) Find all stationary points of f.
b) Compute the Hessian matrix of f. Classify the stationary points of f as local maxima, local minima or saddle points.

13. Mock Final Exam in GRA6035 12/2010, Problem 2

a) Find all stationary points of $f(x,y,z) = e^{xy+yz-xz}$.
b) The function $g(x,y,z) = e^{ax+by+cz}$ is defined on $\mathbb{R}^3$. Determine the values of the parameters a,b,c such that g is convex. Is it concave for any values of a,b,c?

14. Final Exam in GRA6035 30/05/2011, Problem 1
We consider the function $f(x,y,z,w) = x^5 + xy^2 - zw$.

a) Find all stationary points of f.
b) Compute the Hessian matrix of f. Classify the stationary points of f as local maxima, local minima or saddle points.

Solutions

1 The partial derivatives of $f(x,y,z) = x^4 + y^4 + z^4 + x^2 + y^2 + z^2$ are

$$f'_x = 4x^3 + 2x, \quad f'_y = 4y^3 + 2y, \quad f'_z = 4z^3 + 2z$$

The stationary points are given by $2x(2x^2+1) = 2y(2y^2+1) = 2z(2z^2+1) = 0$, and this means that the unique stationary point is $(x,y,z) = (0,0,0)$. The Hessian of f is

$$H(f) = \begin{pmatrix} 12x^2+2 & 0 & 0 \\ 0 & 12y^2+2 & 0 \\ 0 & 0 & 12z^2+2 \end{pmatrix}$$

We see that $H(f)$ is positive definite, and therefore f is convex and $(0,0,0)$ is a global minimum point.

2 The partial derivatives of $f(x,y) = x^3 + y^3 - 3x - 2y$ are

$$f'_x = 3x^2 - 3, \quad f'_y = 3y^2 - 2$$

The stationary points are given by $3x^2 - 3 = 3y^2 - 2 = 0$, and this means that the unique stationary point in S is $(x,y,z) = (1, \sqrt{2/3})$. The Hessian of f is

$$H(f) = \begin{pmatrix} 6x & 0 \\ 0 & 6y \end{pmatrix}$$

We see that $H(f)$ is positive definite since $D_1 = 6x > 0$ and $D_2 = 36xy > 0$, and therefore f is convex and $(1, \sqrt{2/3})$ is a global minimum point.

3 The profit function $\pi(x,y)$ is defined on $\{(x,y) : x > 0,\ y > 0\}$, and is given by

$$\pi(x,y) = 13x + 8y - C(x,y) = -0.04x^2 + 0.01xy - 0.01y^2 + 9x + 6y - 500$$

The Hessian of π is given by

$$H(\pi) = \begin{pmatrix} -0.08 & 0.01 \\ 0.01 & -0.02 \end{pmatrix}$$

and it is negative definite since $D_1 = -0.08 < 0$ and $D_2 = 0.016 - 0.0001 = 0.0159 > 0$, and therefore π is concave. The stationary point of π is given by

$$\pi'_x = -0.08x + 0.01y + 9 = 0, \quad \pi'_y = 0.01x - 0.02y + 6 = 0$$

This gives $(x,y) = (160, 380)$, which is the unique maximum point.

4 To prove that S is a convex set, pick any points $P = (x,y,z)$ and $Q = (x',y',z')$ in S. By definition, $z > 0$ and $z' > 0$, which implies that all points on the line segment $[P,Q]$ have positive z-coordinate as well. This means that $[P,Q]$ is contained in S, and therefore S is convex. The partial derivatives of f are

$$f'_x = 2x + 2y, \quad f'_y = 2x + 2y, \quad f'_z = 3z^2$$

Since $z > 0$, there are no stationary points in S. The Hessian matrix of f is

$$H(f) = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 6z \end{pmatrix}$$

The principal minors are $\Delta_1 = 2, 2, 6z > 0$, $\Delta_2 = 0, 12z, 12z > 0$ and $\Delta_3 = 0$, so $H(f)$ is positive semidefinite and f is convex (but not strictly convex) on S. Since f has no stationary points and S is open (so there are no boundary points), f does not have global extremal points.

5 The partial derivatives of $f(x,y,z) = x^4 + y^4 + z^4 + x^2 - xy + y^2 + yz + z^2$ are

$$f'_x = 4x^3 + 2x - y, \quad f'_y = 4y^3 - x + 2y + z, \quad f'_z = 4z^3 + y + 2z$$

and the Hessian matrix is

$$H(f) = \begin{pmatrix} 12x^2 + 2 & -1 & 0 \\ -1 & 12y^2 + 2 & 1 \\ 0 & 1 & 12z^2 + 2 \end{pmatrix}$$

Since $D_1 = 12x^2 + 2 > 0$, $D_2 = (12x^2 + 2)(12y^2 + 2) - 1 = 144x^2y^2 + 24x^2 + 24y^2 + 3 > 0$ and $D_3 = -1(12x^2 + 2) + (12z^2 + 2)D_2 = 1728x^2y^2z^z + 288(x^2y^2 + x^2z^2 + y^2z^2) + 36x^2 + 48y^2 + 36z^2 + 4 > 0$, we see that f is convex.

6 The partial derivatives of the function $f(x,y,z) = -2x^4 + 2yz - y^2 + 8x$ is

$$f'_x = -8x^3 + 8, \quad f'_y = 2z - 2y, \quad f'_z = 2y$$

Hence the stationary points are given by $y = 0, z = 0, x = 1$ or $(x,y,z) = (1,0,0)$. The Hessian matrix of f is

$$H(f) = \begin{pmatrix} -24x^2 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & 0 \end{pmatrix} \quad \Rightarrow \quad H(f)(1,0,0) = \begin{pmatrix} -24 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & 0 \end{pmatrix}$$

Since $D_1 = -24 < 0$, $D_2 = 48 > 0$, but $D_3 = 96 > 0$, we see that the stationary point $(1,0,0)$ is a saddle point.

7 The Hessian matrix of the function $f(x,y,z) = x^2 + y^2 + 3z^2 - xy + 2xz + yz$ is

$$H(f) = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & 1 \\ 2 & 1 & 6 \end{pmatrix}$$

Since $D_1 = 2 > 0$, $D_2 = 3 > 0$, $D_3 = 2(-5) - 1(4) + 6D_2 = 4 > 0$, we see that $H(f)$ is positive definite, and that the unique stationary point is a local minimum point.

8 The partial derivatives of the function $f(x,y) = x^3 - 3xy + y^3$ are

$$f'_x = 3x^2 - 3y, \quad f'_y = -3x + 3y^2$$

The stationary points are therefore given by $3x^2 - 3y = 0$ or $y = x^2$, and $-3x + 3y^2 = 0$ or $y^2 = x^4 = x$. This gives $x = 0$ or $x^3 = 1$, that is, $x = 1$. The stationary points are $(x,y) = (0,0), (1,1)$. The Hessian matrix of f is

$$H(f) = \begin{pmatrix} 6x & -3 \\ -3 & 9y \end{pmatrix} \quad \Rightarrow \quad H(f)(0,0) = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}, \; H(f)(1,1) = \begin{pmatrix} 6 & -3 \\ -3 & 9 \end{pmatrix}$$

In the first case, $D_1 = 0$; $D_2 = -9 < 0$ so $(0,0)$ is a saddle point. In the second case, $D_1 = 6$, $D_2 = 45 > 0$, so $(1,1)$ is a local minimum point.

9 The partial derivatives of the function $f(x,y,z) = x^3 + 3xy + 3xz + y^3 + 3yz + z^3$ are

$$f'_x = 3x^2 + 3y + 3z, \quad f'_y = 3x + 3y^2 + 3z, \; f'_z = 3x + 3y + 3z^2$$

The stationary points are given by $x^2 + y + z = 0$, $x + y^2 + z = 0$ and $x + y + z^2 = 0$. The first equation gives $z = -x^2 - y$, and the second becomes $x + y^2 + (-x^2 - y) = 0$, or $x - y = x^2 - y^2 = (x-y)(x+y)$. This implies that $x - y = 0$ or that $x + y = 1$. We see that $x + y = 1$ implies that $1 + z^2 = 0$ from the third equation, and this is impossible, and we infer that $x - y = 0$, or $x = y$. Then $z = -x^2 - x$ from the computation above, and the last equation gives

$$x + y + z^2 = 2x + (-x^2 - x)^2 = x^4 + 2x^3 + x^2 + 2x = (x+2)(x^3 + x) = 0$$

Hence $x = 0$, $x = -2$ or $x^2 + 1 = 0$. The last equation has not solutions, se we get two stationary points $(x,y,z) = (0,0,0), (-2,-2,-2)$. The Hessian matrix of f at $(0,0,0)$ is

$$H(f) = \begin{pmatrix} 6x & 3 & 3 \\ 3 & 6y & 3 \\ 3 & 3 & 6z \end{pmatrix} \quad \Rightarrow \quad H(f)(0,0,0) = \begin{pmatrix} 0 & 3 & 3 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{pmatrix}$$

In this case, $D_1 = 0$; $D_2 = -9 < 0$, so $(0,0,0)$ is a saddle point. At $(-2,-2,-2)$, the Hessian is

$$H(f) = \begin{pmatrix} 6x & 3 & 3 \\ 3 & 6y & 3 \\ 3 & 3 & 6z \end{pmatrix} \quad \Rightarrow \quad H(f)(-2,-2,-2) = \begin{pmatrix} -12 & 3 & 3 \\ 3 & -12 & 3 \\ 3 & 3 & -12 \end{pmatrix}$$

In this case, $D_1 = -12$, $D_2 = 135 > 0$, $D_3 = -50 < 0$, so $(-2,-2,-2)$ is a local maximum point.

10 We consider the Lagrangian $\mathscr{L}(x,y,z,\lambda) = 100 - x^2 - y^2 - z^2 - \lambda(x+2y+z)$, and solve the first order conditions

$$\begin{aligned} \mathscr{L}'_x &= -2x - \lambda = 0 \\ \mathscr{L}'_y &= -2y - \lambda \cdot 2 = 0 \\ \mathscr{L}'_z &= -2z - \lambda = 0 \end{aligned}$$

together with $x+2y+z = a$. We get $2x = -\lambda,\ 2y = -2\lambda,\ 2z = -\lambda$ and (after multiplying the constraint by 2)

$$-\lambda - 4\lambda - \lambda = 2a \quad \Rightarrow \quad \lambda = -a/3$$

The unique solution of the equations is $(x,y,z;\lambda) = (a/6, a/3, a/6; -a/3)$. Since $\mathscr{L}(x,y,z;-a/3)$ is a concave function in (x,y,z), we have that this solution solves the maximum problem. The optimal value function

$$f^*(a) = f(a/6, a/3, a/6) = 100 - \frac{a^2}{36} - \frac{a^2}{9} - \frac{a^2}{36} = 100 - \frac{a^2}{6}$$

We see that the derivative of the optimal value function is $-2a/6 = -a/3 = \lambda(a)$.

11 We consider the Lagrangian

$$\mathscr{L}(x,y,z,\lambda_1,\lambda_2) = x + 4y + z - \lambda_1(x^2+y^2+z^2) - \lambda_2(x+2y+3z)$$

and solve the first order conditions

$$\begin{aligned} \mathscr{L}'_x &= 1 - \lambda_1 \cdot 2x - \lambda_2 = 0 \\ \mathscr{L}'_y &= 4 - \lambda_1 \cdot 2y - \lambda_2 \cdot 2 = 0 \\ \mathscr{L}'_z &= 1 - \lambda_1 \cdot 2z - \lambda_2 \cdot 3 = 0 \end{aligned}$$

together with $x^2+y^2+z^2 = 216$ and $x+2y+3z = 0$. From the first order conditions, we get

$$2x\lambda_1 = 1 - \lambda_2,\ 2y\lambda_1 = 4 - 2\lambda_2,\ 2z\lambda_1 = 1 - 3\lambda_2$$

We see from these equations that we cannot have $\lambda_1 = 0$, and multiply the last constraint with $2\lambda_1$. We get

$$2\lambda_1(x+2y+3z) = 0 \quad \Rightarrow \quad (1-\lambda_2) + 2(4-2\lambda_2) + 3(1-3\lambda_2) = 0$$

This gives $12 - 14\lambda_2 = 0$, or $\lambda_2 = 12/14 = 6/7$. We use this and solve for x, y, z, and get

$$x = \frac{1}{14\lambda_1},\ y = \frac{8}{7\lambda_1},\ z = -\frac{11}{14\lambda_1}$$

Then we substitute this in the first constraint, and get

$$\left(\frac{1}{14\lambda_1}\right)^2 (1+16^2+(-11)^2) = 216 \quad \Rightarrow \quad 216\cdot 14^2\lambda_1^2 = 378$$

This implies that $\lambda_1 = \pm\frac{\sqrt{7}}{28}$, and we have two solutions to the first order equations and constraints. Moreover, we see that $\mathscr{L}(x,y,z,\pm\frac{\sqrt{7}}{28},\frac{6}{7})$ is a concave function in (x,y,z) when $\lambda_1 > 0$, and convex when $\lambda_1 < 0$. Therefore, the solution

$$(x^*,y^*,z^*) = (\frac{2}{7}\sqrt{7}, \frac{32}{7}\sqrt{7}, -\frac{22}{7}\sqrt{7})$$

corresponding to $\lambda_1 = \frac{\sqrt{7}}{28}$ solves the maximum problem, and the maximum value is

$$f(x^*,y^*,z^*) = x^*+4y^*+z^* = \frac{2+128-22}{7}\sqrt{7} = \frac{108}{7}\sqrt{7} \simeq 40.820$$

When $b_1 = 216$ is changed to 215 and $b_2 = 0$ is changed to 0.1, the approximate change in the the maximum value is given by

$$\lambda_1(215-216)+\lambda_2(0.1-0) = (-1)\frac{\sqrt{7}}{28}+(0.1)\frac{6}{7} \simeq -0.009$$

The estimate for the new maximum value is therefore $\simeq 40.811$.

12 Final Exam in GRA6035 10/12/2010, Problem 1

a) We compute the partial derivatives $f'_x = (x^2+2x)e^x$, $f'_y = z$ and $f'_z = y-3z^2$. The stationary points are given by the equations

$$(x^2+2x)e^x = 0, \quad z = 0, \quad y-3z^2 = 0$$

and this gives $x = 0$ or $x = -2$ from the first equation and $y = 0$ and $z = 0$ from the last two. The stationary points are therefore $(x,y,z) = (\mathbf{0},\mathbf{0},\mathbf{0}),(-\mathbf{2},\mathbf{0},\mathbf{0})$.

b) We compute the second order partial derivatives of f and form the Hessian matrix

$$f'' = \begin{pmatrix} (x^2+4x+2)e^x & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -6z \end{pmatrix}$$

We see that the second order principal minor obtained from the last two rows and columns is

$$\begin{vmatrix} 0 & 1 \\ 1 & -6z \end{vmatrix} = -1 < 0$$

hence the Hessian is indefinite in all stationary points. Therefore, both stationary points are **saddle points**.

13 Mock Final Exam in GRA6035 12/2010, Problem 2

a) We write $f(x,y,z) = e^u$ with $u = xy + yz - xz$, and compute

$$f'_x = e^u(y-z),\ f'_y = e^u(x+z),\ f'_z = e^u(y-x)$$

The stationary points of f are therefore given by

$$y - z = 0,\ x + z = 0,\ y - x = 0$$

which gives $(x,y,z) = (0,0,0)$. This is the unique stationary points of f.

b) We write $f(x,y,z) = e^u$ with $u = ax + by + cz$, and compute that

$$g'_x = e^u \cdot a,\ g'_y = e^u \cdot b,\ g'_z = e^u \cdot c$$

and this gives Hessian matrix

$$H(g) = \begin{pmatrix} a^2e^u & abe^u & ace^u \\ abe^u & b^2e^u & bce^u \\ ace^u & bce^u & c^2e^u \end{pmatrix} = e^u \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix}$$

This gives principal minors $\Delta_1 = e^u a^2, e^u b^2, e^u c^2 \geq 0$, $\Delta_2 = 0,0,0$ and $\Delta_3 = 0$. Hence g is convex for all values of a,b,c, and g is concave if and only if $a = b = c = 0$.

14 Final Exam in GRA6035 30/05/2011, Problem 1

a) We compute the partial derivatives $f'_x = 5x^4 + y^2$, $f'_y = 2xy$, $f'_z = -w$ and $f'_w = -z$. The stationary points are given by

$$5x^4 + y^2 = 0, \quad 2xy = 0, \quad -w = 0, \quad -z = 0$$

and this gives $z = w = 0$ from the last two equations, and $x = y = 0$ from the first two. The stationary points are therefore $(x,y,z,w) = (\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0})$.

b) We compute the second order partial derivatives of f and form the Hessian matrix

$$f'' = \begin{pmatrix} 20x^3 & 2y & 0 & 0 \\ 2y & 2x & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

We see that the second order principal minor obtained from the last two rows and columns is

$$\begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} = -1 < 0$$

hence the Hessian is indefinite. Therefore, the stationary point is a **saddle point**.