Problem Sheet 5 with Solutions
GRA 6035 Mathematics

BI Norwegian Business School

Problems

1. Find the symmetric matrix of the following quadratic forms:

a) $Q(x,y) = x^2 + 2xy + y^2$
b) $Q(x,y) = ax^2 + bxy + cy^2$
c) $Q(x,y,z) = 3x^2 - 2xy + 3xz + 2y^2 + 3z^2$

2. Find the symmetric matrix and determine the definiteness of the following quadratic forms:

a) $Q(\mathbf{x}) = x_1^2 + 3x_2^2 + 5x_3^2$
b) $Q(\mathbf{x}) = x_1^2 + 2x_1x_2 + 3x_2^2 + 5x_3^2$

3. Compute all leading principal minors and all principal minors of the following matrices:

$$a) \quad A = \begin{pmatrix} -3 & 4 \\ 4 & -5 \end{pmatrix} \qquad\qquad b) \quad A = \begin{pmatrix} -3 & 4 \\ 4 & -6 \end{pmatrix}$$

In each case, write down the corresponding quadratic form $Q(x,y) = \mathbf{x}^T A\mathbf{x}$, and determine its definiteness. Use this to classify the stationary point $(x,y) = (0,0)$ of the quadratic form.

4. Compute all leading principal minors and all principal minors of the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{pmatrix}$$

Write down the corresponding quadratic form $Q(x,y,z) = \mathbf{x}^T A\mathbf{x}$, and determine its definiteness. Use this to classify the stationary point $(x,y,z) = (0,0,0)$ of the quadratic form.

5. For which values of the parameters a,b,c is the symmetric matrix

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

positive (semi)definite and negative (semi)definite?

6. Determine the definiteness of the following constrained quadratic forms using bordered Hessians:

a) $Q(x,y) = x^2 + 2xy - y^2$ subject to $x + y = 0$
b) $Q(\mathbf{x}) = x_1^2 + x_2^2 - x_3^2 + 4x_1x_3 - 2x_1x_2$ subject to $x_1 + x_2 + x_3 = 0$, $x_1 + x_2 - x_3 = 0$

7. Determine the definiteness of the following constrained quadratic forms without using bordered Hessians:

a) $Q(x,y) = x^2 + 2xy - y^2$ subject to $x + y = 0$

b) $Q(x_1,x_2,x_3) = x_1^2 + x_2^2 - x_3^2 + 4x_1x_3 - 2x_1x_2$ subject to $x_1 + x_2 + x_3 = 0$ and $x_1 + x_2 - x_3 = 0$

8. Midterm in GRA6035 24/09/2010, Problem 6
Consider the quadratic form

$$Q(x_1,x_2) = x_1^2 - 4x_1x_2 + 4x_2^2$$

Which statement is true?

a) Q is positive semidefinite but not positive definite
b) Q is negative semidefinite but not negative definite
c) Q is indefinite
d) Q is positive definite
e) I prefer not to answer.

9. Mock Midterm in GRA6035 09/2010, Problem 6
Consider the function

$$f(x_1,x_2,x_3) = x_1^2 + 6x_1x_2 + 3x_2^2 + 2x_3^2$$

Which statement is true?

a) f is not a quadratic form
b) f is a positive definite quadratic form
c) f is an indefinite quadratic form
d) f is a negative definite quadratic form
e) I prefer not to answer.

10. Midterm in GRA6035 24/05/2011, Problem 6
Consider the quadratic form

$$Q(x_1,x_2) = -2x_1^2 + 12x_1x_2 + 2x_2^2$$

Which statement is true?

a) Q is positive semidefinite but not positive definite
b) Q is negative semidefinite but not negative definite
c) Q is indefinite
d) Q is positive definite
e) I prefer not to answer.

Solutions

1 The symmetric matrices are given by

$$a) \quad A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad b) \quad A = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \qquad c) \quad A = \begin{pmatrix} 3 & -1 & 3/2 \\ -1 & 2 & 0 \\ 3/2 & 0 & 3 \end{pmatrix}$$

2 The symmetric matrices are given by

$$a) \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \qquad b) \quad A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

In a) the eigenvalues $\lambda = 1, 3, 5$ are all positive, so the quadratic form Q and the matrix A are positive definite. In b), we compute the eigenvalues using the characteristic equation

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 3-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix} = (5-\lambda)(\lambda^2 - 4\lambda + 2) = 0$$

and get $\lambda = 5$ and $\lambda = 2 \pm \sqrt{2}$. Since all eigenvalues are positive, the quadratic form Q and the matrix A are positive definite. Another way to determine the definiteness is to compute the leading principal minors $D_1 = 1$, $D_2 = 2$, and $D_3 = 10$.

3 In a) the leading principal minors are $D_1 = -3$ and $D_2 = -1$, and the principal minors are $\Delta_1 = -3, -5$ and $\Delta_2 = -1$. From the leading principal minors, we see that A is indefinite, and this means that $(x, y) = (0, 0)$ is a saddle point for the quadratic form. In b) the leading principal minors are $D_1 = -3$ and $D_2 = 2$, and the principal minors are $\Delta_1 = -3, -6$ and $\Delta_2 = 2$. From the leading principal minors, we see that A is negative definite, and this means that $(x, y) = (0, 0)$ is a global maximum for the quadratic form.

4 The leading principal minors and principal minors of the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{pmatrix}$$

are given by

$$D_1 = 1,\ D_2 = 0,\ D_3 = |A| = 1(24-25) - 2(12) = -25$$

and

$$\Delta_1 = 1, 4, 6,\ \Delta_2 = 0, 6, -1,\ \Delta_3 = -25$$

The quadratic form of A is given by $Q(x,y,z) = x^2 + 4xy + 4y^2 + 10yz + 6z^2$. Since there is a principal minor $\Delta_2 = -1$ of order two that is negative, the quadratic form is indefinite, and the stationary point $(x,y,z) = (0,0,0)$ is a saddle point.

5 The leading principal minors are given by $D_1 = a$, $D_2 = ac - b^2$ and the principal minors are given by $\Delta_1 = a, c$, $\Delta_2 = ac - b^2$. Hence we have that

$$\text{positive definite} \Leftrightarrow a > 0, ac - b^2 > 0$$
$$\text{negative definite} \Leftrightarrow a < 0, ac - b^2 > 0$$
$$\text{positive semidefinite} \Leftrightarrow a \geq 0, c \geq 0, ac - b^2 \geq 0$$
$$\text{negative semidefinite} \Leftrightarrow a \leq 0, c \leq 0, ac - b^2 \geq 0$$

6 In a) we have $n = 2$ and $m = 1$, so $n - m = 1$. We consider the determinant of the bordered Hessian

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = -1(-2) + 1(0) = 2$$

Since it has the same sign as $(-1)^n = (-1)^2 = 1$, we see that the constrained quadratic form is negative definite. In b) we have $n = 3$ and $m = 2$, so $n - m = 1$. We consider the determinant of the bordered Hessian

$$\begin{vmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 2 \\ 1 & 1 & -1 & 1 & 0 \\ 1 & -1 & 2 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -2 \\ 1 & 1 & 1 & -1 & 2 \\ 0 & 0 & -2 & 2 & -2 \\ 0 & -2 & 1 & 1 & -3 \end{vmatrix} = 2 \begin{vmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 0 & 0 & -2 & 2 \\ 0 & -2 & 1 & 1 \end{vmatrix} = 2(-1)(-2)(4) = 16$$

Since it has the same sign as $(-1)^m = (-1)^2 = 1$, we see that the constrained quadratic form is positive definite.

7 In a) we solve the constraint and get $y = -x$. Substitution in the quadratic form gives $Q(x,y) = x^2 + 2xy - y^2 = -2x^2$, which is negative definite. Therefore, the constrained quadratic form in a) is negative definite. In b) we solve the constraint using Gaussian elimination and get one free variable x_2 and solution $x_1 = -x_2$ and $x_3 = 0$. Substitution in the quadratic form gives $Q(x_1,x_2,x_3) = x_1^2 + x_2^2 - x_3^2 + 4x_1x_3 - 2x_1x_2 = 4x_2^2$, which is positive definite. Therefore, the constrained quadratic form in b) is positive definite.

8 Midterm in GRA6035 24/09/2010, Problem 6
The symmetric matrix associated with Q is

$$A = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$$

We compute its eigenvalues to be 0 and 5. Hence the correct answer is alternative **A**. This question can also be answered using the fact that the principal minors are $1,4$ (of order one) and 0 (of order two), or the fact that $Q(x_1,x_2) = (x_1 - 2x_2)^2$.

9 Mock Midterm in GRA6035 09/2010, Problem 6

Since all terms of f have degree two, it is a quadratic form, and its symmetric matrix is

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

The characteristic equation of A is $(\lambda^2 - 4\lambda - 6)(2 - \lambda) = 0$, and the eigenvalues are $\lambda = 2$ and $\lambda = 2 \pm \sqrt{10}$. Hence the correct answer is alternative **C**. This problem can also be solved using the principal leading minors, which are $D_1 = 1$, $D_2 = -6$ and $D_3 = -12$.

10 Midterm in GRA6035 24/05/2011, Problem 6

The symmetric matrix associated with Q is

$$A = \begin{pmatrix} -2 & 6 \\ 6 & 2 \end{pmatrix}$$

We compute its eigenvalues to be $\pm\sqrt{40}$. Hence the correct answer is alternative **C**. This problem can also be solved using the principal leading minors, which are $D_1 = -2$ and $D_2 = -40$.