Problem Sheet 2 with Solutions
GRA 6035 Mathematics

BI Norwegian Business School

Problems

1. Compute $4A+2B$, AB, BA, BI and IA when

$$A=\begin{pmatrix}2&3\\4&1\end{pmatrix},\quad B=\begin{pmatrix}2&6\\7&0\end{pmatrix},\quad I=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

2. One of the laws of matrix algebra states that $(AB)^T=B^TA^T$. Prove this when A and B are 2×2-matrices.

3. Simplify the following matrix expressions:

$$\begin{aligned}&a)\quad AB(BC-CB)+(CA-AB)BC+CA(A-B)C\\&b)\quad (A-B)(C-A)+(C-B)(A-C)+(C-A)^2\end{aligned}$$

4. A general $m\times n$-matrix is often written $A=(a_{ij})_{m\times n}$, where a_{ij} is the entry of A in row i and column j. Prove that if $m=n$ and $a_{ij}=a_{ji}$ for all i and j, then $A=A^T$. Give a concrete example of a matrix with this property, and explain why it is reasonable to call a matrix A symmetric when $A=A^T$.

5. Compute D^2, D^3 and D^n when

$$D=\begin{pmatrix}2&0&0\\0&-3&0\\0&0&-1\end{pmatrix}$$

6. Write down the 3×3 linear system corresponding to the matrix equation $A\mathbf{x}=\mathbf{b}$ when

$$A=\begin{pmatrix}3&1&5\\5&-3&2\\4&-3&-1\end{pmatrix},\quad \mathbf{x}=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix},\quad \mathbf{b}=\begin{pmatrix}4\\-2\\-1\end{pmatrix}$$

7. Initially, three firms A, B and C (numbered 1, 2 and 3) share the market for a certain commodity. Firm A has 20% of the marked, B has 60% and C has 20%. In course of the next year, the following changes occur:

A keeps 85% of its customers, while losing 5% to B and 10% to C
B keeps 55% of its customers, while losing 10% to A and 35% to C
C keeps 85% of its customers, while losing 10% to A and 5% to B

We can represent market shares of the three firms by means of *a market share vector*, defined as a column vector $\mathbf{s}$ whose components are all non-negative and sum to 1. Define the matrix $\mathbf{T}$ and the initial share vector $\mathbf{s}$ by

$$T=\begin{pmatrix}0.85&0.10&0.10\\0.05&0.55&0.05\\0.10&0.35&0.85\end{pmatrix},\quad \mathbf{s}=\begin{pmatrix}0.2\\0.6\\0.2\end{pmatrix}$$

The matrix T is called the *transition matrix*. Compute the vector $T\mathbf{s}$, show that it is also a market share vector, and give an interpretation. What is the interpretation of $T^2\mathbf{s}$ and $T^3\mathbf{s}$? Finally, compute $T\mathbf{q}$ when

$$\mathbf{q} = \begin{pmatrix} 0.4 \\ 0.1 \\ 0.5 \end{pmatrix}$$

and give an interpretation.

8. Compute the following matrix product using partitioning. Check the result by ordinary matrix multiplication:

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & 0 & -1 \end{array}\right) \cdot \left(\begin{array}{cc} 2 & -1 \\ 0 & 1 \\ \hline 1 & 1 \end{array}\right)$$

9. If $A = (a_{ij})_{n\times n}$ is an $n \times n$-matrix, then its determinant may be computed by

$$|A| = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

where C_{ij} is the cofactor in position (i, j). This is called cofactor expansion along the first row. Similarly one may compute $|A|$ by cofactor expansion along any row or column. Compute $|A|$ using cofactor expansion along the first column, and then along the third row, when

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 1 & 0 & 8 \end{pmatrix}$$

Check that you get the same answer. Is A invertible?

10. Let A and B be 3×3-matrices with $|A| = 2$ and $|B| = -5$. Find $|AB|$, $|-3A|$ and $|-2A^T|$. Compute $|C|$ when C is the matrix obtained from B by interchanging two rows.

11. Compute the determinant using elementary row operations:

$$\begin{vmatrix} 3 & 1 & 5 \\ 9 & 3 & 15 \\ -3 & -1 & -5 \end{vmatrix}$$

12. Without computing the determinants, show that

$$\begin{vmatrix} b^2+c^2 & ab & ac \\ ab & a^2+c^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix} = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2$$

13. Find the inverse matrix A^{-1}, if it exists, when A is the matrix given by

$$a)\quad A=\begin{pmatrix}1&3\\1&3\end{pmatrix}\qquad b)\quad A=\begin{pmatrix}1&3\\-1&3\end{pmatrix}\qquad c)\quad A=\begin{pmatrix}1&2\\0&1\end{pmatrix}$$

14. Compute the cofactor matrix, the adjoint matrix and the inverse matrix of these matrices:

$$a)\quad A=\begin{pmatrix}1&2&3\\0&5&6\\1&0&8\end{pmatrix}\qquad b)\quad B=\begin{pmatrix}1&0&b\\0&1&0\\0&0&1\end{pmatrix}$$

Verify that $AA^{-1}=I$ and that $BB^{-1}=I$.

15. Write the linear system of equations

$$\begin{aligned}5x_1+x_2&=3\\2x_1-x_2&=4\end{aligned}$$

on matrix form $A\mathbf{x}=\mathbf{b}$ and solve it using A^{-1}.

16. There is an efficient way of finding the inverse of a square matrix using row operations. Suppose we want to find the inverse of

$$A=\begin{pmatrix}1&2&3\\1&3&3\\2&5&7\end{pmatrix}$$

To do this we form the partitioned matrix

$$\big(A\big|I\big)=\left(\begin{array}{ccc|ccc}1&2&3&1&0&0\\1&3&3&0&1&0\\2&5&7&0&0&1\end{array}\right)$$

and then reduced it to reduced echelon form using elementary row operations: First, we add (-1) times the first row to the second row

$$\left(\begin{array}{ccc|ccc}1&2&3&1&0&0\\1&3&3&0&1&0\\2&5&7&0&0&1\end{array}\right)\quad\Rightarrow\quad\left(\begin{array}{ccc|ccc}1&2&3&1&0&0\\0&1&0&-1&1&0\\2&5&7&0&0&1\end{array}\right)$$

Then we add (-2) times the first row to the last row

$$\left(\begin{array}{ccc|ccc}1&2&3&1&0&0\\0&1&0&-1&1&0\\2&5&7&0&0&1\end{array}\right)\quad\Rightarrow\quad\left(\begin{array}{ccc|ccc}1&2&3&1&0&0\\0&1&0&-1&1&0\\0&1&1&-2&0&1\end{array}\right)$$

Then we add (-1) times the second row to the third

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right)$$

Next, we add (-3) times the last row to the first

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 4 & 3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right)$$

Then we add (-2) times the second row to the first

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 4 & 3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & 1 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right)$$

We now have the partitioned matrix $(I|A^{-1})$ and thus

$$A^{-1} = \begin{pmatrix} 6 & 1 & -3 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

Use the same technique to find the inverse of the following matrices:

$$a) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad b) \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad c) \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad d) \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

17. Describe all minors of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}$$

It is not necessary to compute all the minors.

18. Determine the ranks of these matrices for all values of the parameters:

$$a) \begin{pmatrix} x & 0 & x^2-2 \\ 0 & 1 & 1 \\ -1 & x & x-1 \end{pmatrix} \qquad b) \begin{pmatrix} t+3 & 5 & 6 \\ -1 & t-3 & -6 \\ 1 & 1 & t+4 \end{pmatrix}$$

19. Give an example where $\mathrm{rk}(AB) \neq \mathrm{rk}(BA)$. Hint: Try some 2×2 matrices.

20. Use minors to determine if the systems have solutions. If they do, determine the number of degrees of freedom. Find all solutions and check the results.

$$a)\quad \begin{aligned} -2x_1 - 3x_2 + x_3 &= 3 \\ 4x_1 + 6x_2 - 2x_3 &= 1 \end{aligned} \qquad b)\quad \begin{aligned} x_1 + x_2 - x_3 + x_4 &= 2 \\ 2x_1 - x_2 + x_3 - 3x_4 &= 1 \end{aligned}$$

$$c)\quad \begin{aligned} x_1 - x_2 + 2x_3 + x_4 &= 1 \\ 2x_1 + x_2 - x_3 + 3x_4 &= 3 \\ x_1 + 5x_2 - 8x_3 + x_4 &= 1 \\ 4x_1 + 5x_2 - 7x_3 + 7x_4 &= 7 \end{aligned} \qquad d)\quad \begin{aligned} x_1 + x_2 + 2x_3 + x_4 &= 5 \\ 2x_1 + 3x_2 - x_3 - 2x_4 &= 2 \\ 4x_1 + 5x_2 + 3x_3 \phantom{{}+x_4} &= 7 \end{aligned}$$

21. Let $A\mathbf{x} = \mathbf{b}$ be a linear system of equations in matrix form. Prove that if $\mathbf{x}_1$ and $\mathbf{x}_2$ are both solutions of the system, then so is $\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2$ for every number λ. Use this fact to prove that a linear system of equations that is consistent has either one solution or infinitely many solutions.

22. Find the rank of A for all values of the parameter t, and solve $A\mathbf{x} = \mathbf{b}$ when $t = -3$:

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 5 & t \\ 4 & 7-t & -6 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 11 \\ 3 \\ 6 \end{pmatrix}$$

23. Midterm Exam in GRA6035 24/09/2010, Problem 1
Consider the linear system

$$\begin{pmatrix} 1 & -3 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 4 \\ 0 \end{pmatrix}$$

Which statement is true?

a) The linear system is inconsistent.
b) The linear system has a unique solution.
c) The linear system has one degree of freedom
d) The linear system has two degrees of freedom
e) I prefer not to answer.

24. Mock Midterm Exam in GRA6035 09/2010, Problem 1
Consider the linear system

$$\begin{pmatrix} 3 & -9 & 12 & -9 & 0 \\ 0 & 2 & -4 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -9 \\ -14 \\ 4 \\ 7 \end{pmatrix}$$

Which statement is true?

a) The linear system has a unique solution.
b) The linear system has one degree of freedom
c) The linear system has two degrees of freedom
d) The linear system is inconsistent.
e) I prefer not to answer.

25. Midterm Exam in GRA6035 24/05/2011, Problem 3
Consider the linear system

$$\begin{pmatrix} 1 & 2 & -3 & -1 & 0 \\ 0 & 1 & 7 & 3 & -4 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 2 \\ 1 \end{pmatrix}$$

Which statement is true?

a) The linear system is inconsistent
b) The linear system has a unique solution
c) The linear system has one degree of freedom
d) The linear system has two degrees of freedom
e) I prefer not to answer.

Solutions

1 We have

$$4A+2B=\begin{pmatrix}12&24\\30&4\end{pmatrix},\quad AB=\begin{pmatrix}25&12\\15&24\end{pmatrix},\quad BA=\begin{pmatrix}28&12\\14&21\end{pmatrix},\quad BI=B,\quad IA=A$$

2 Let

$$A=\begin{pmatrix}a&b\\c&d\end{pmatrix},\quad B=\begin{pmatrix}x&y\\z&w\end{pmatrix}$$

Then we have

$$AB=\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}x&y\\z&w\end{pmatrix}=\begin{pmatrix}ax+bz&bw+ay\\cx+dz&dw+cy\end{pmatrix}\implies(AB)^T=\begin{pmatrix}ax+bz&cx+dz\\bw+ay&dw+cy\end{pmatrix}$$

and

$$A^T=\begin{pmatrix}a&c\\b&d\end{pmatrix},\quad B^T=\begin{pmatrix}x&z\\y&w\end{pmatrix}\implies B^TA^T=\begin{pmatrix}ax+bz&cx+dz\\bw+ay&dw+cy\end{pmatrix}$$

Comparing the expressions, we see that $(AB)^T=B^TA^T$.

3 We have

$$\begin{aligned}(a)\quad & AB(BC-CB)+(CA-AB)BC+CA(A-B)C=ABBC-ABCB+CABC\\ & -ABBC+CAAC-CABC=-ABCB+CAAC=-ABCB+CA^2C\\(b)\quad & (A-B)(C-A)+(C-B)(A-C)+(C-A)^2=AC-A^2-BC+BA+CA\\ & -C^2-BA+BC+C^2-CA-AC+A^2=0\end{aligned}$$

4 The entry in position (j,i) in A^T equals the entry in position (i,j) in A. Therefore, a square matrix A satisfies $A^T=A$ if $a_{ij}=a_{ji}$. The matrix

$$A=\begin{pmatrix}13&3&2\\3&-2&4\\2&4&3\end{pmatrix}$$

has this property. The condition that $a_{ij}=a_{ji}$ is a symmetry along the diagonal of A, so it is reasonable to call a matrix with $A^T=A$ symmetric.

5 We compute

$$D^2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{pmatrix}^2 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D^3 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{pmatrix}^3 = \begin{pmatrix} 8 & 0 & 0 \\ 0 & -27 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$D^n = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{pmatrix}^n = \begin{pmatrix} 2^n & 0 & 0 \\ 0 & (-3)^n & 0 \\ 0 & 0 & (-1)^n \end{pmatrix}$$

6 We compute

$$A\mathbf{x} = \begin{pmatrix} 3 & 1 & 5 \\ 5 & -3 & 2 \\ 4 & -3 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1 + x_2 + 5x_3 \\ 5x_1 - 3x_2 + 2x_3 \\ 4x_1 - 3x_2 - x_3 \end{pmatrix}$$

Thus we see that $A\mathbf{x} = \mathbf{b}$ if and only if

$$\begin{aligned} 3x_1 + x_2 + 5x_3 &= 4 \\ 5x_1 - 3x_2 + 2x_3 &= -2 \\ 4x_1 - 3x_2 - x_3 &= -1 \end{aligned}$$

7 We compute

$$T\mathbf{s} = \begin{pmatrix} 0.85 & 0.10 & 0.10 \\ 0.05 & 0.55 & 0.05 \\ 0.10 & 0.35 & 0.85 \end{pmatrix} \begin{pmatrix} 0.2 \\ 0.6 \\ 0.2 \end{pmatrix} = \begin{pmatrix} 0.25 \\ 0.35 \\ 0.4 \end{pmatrix}$$

This vector is a market share vector since $0.25 + 0.35 + 0.4 = 1$, and it represents the market shares after one year. We have $T^2\mathbf{s} = T(T\mathbf{s})$ and $T^3\mathbf{s} = T(T^2\mathbf{s})$, so these vectors are the marked share vectors after two and three years. Finally, we compute

$$T\mathbf{q} = \begin{pmatrix} 0.85 & 0.10 & 0.10 \\ 0.05 & 0.55 & 0.05 \\ 0.10 & 0.35 & 0.85 \end{pmatrix} \begin{pmatrix} 0.4 \\ 0.1 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.1 \\ 0.5 \end{pmatrix}$$

We see that $T\mathbf{q} = \mathbf{q}$; if the market share vector is $\mathbf{q}$, then it does not change. Hence $\mathbf{q}$ is an *equilibrium*.

8 We write the matrix product as

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & 0 & -1 \end{array}\right) \cdot \left(\begin{array}{cc} 2 & -1 \\ 0 & 1 \\ \hline 1 & 1 \end{array}\right) = \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = AC + BD$$

We compute

$$AC = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -2 & 1 \end{pmatrix}, \quad BD = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

Hence, we get

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & 0 & -1 \end{array}\right) \cdot \left(\begin{array}{cc} 2 & -1 \\ 0 & 1 \\ \hline 1 & 1 \end{array}\right) = \begin{pmatrix} 2 & 0 \\ -2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -3 & 0 \end{pmatrix}$$

Ordinary matrix multiplication gives the same result.

9 We first calculate $|A|$ using cofactor expansion along the first column:

$$\begin{aligned} |A| &= a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} \\ &= (-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} 5 & 6 \\ 0 & 8 \end{vmatrix} + (-1)^{2+1} \cdot 0 \cdot \begin{vmatrix} 2 & 3 \\ 0 & 8 \end{vmatrix} + (-1)^{3+1} \cdot 1 \cdot \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} \\ &= (5 \cdot 8 - 0 \cdot 6) + 0 + (2 \cdot 6 - 5 \cdot 3) \\ &= 40 + 12 - 15 = 37 \end{aligned}$$

We then calculate $|A|$ using cofactor expansion along the third row:

$$\begin{aligned} |A| &= a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} \\ &= (-1)^{3+1} \cdot 1 \cdot \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} + (-1)^{3+2} \cdot 0 \cdot \begin{vmatrix} 1 & 3 \\ 0 & 6 \end{vmatrix} + (-1)^{3+3} \cdot 8 \cdot \begin{vmatrix} 1 & 2 \\ 0 & 5 \end{vmatrix} \\ &= (2 \cdot 6 - 5 \cdot 3) + 0 + 8 \cdot (1 \cdot 5 - 0 \cdot 2) \\ &= 12 - 15 + 8 \cdot 5 = 37 \end{aligned}$$

We see that $\det(A) = 37 \neq 0$ using both methods, hence A is invertible.

10 We compute

$$\begin{aligned} |AB| &= |A||B| = 2 \cdot (-5) = -10 \\ |-3A| &= (-3)^3|A| = (-27) \cdot 2 = -54 \\ |-2A^T| &= (-2)^3|A^T| = (-8) \cdot |A| = (-8) \cdot 2 = -16 \\ |C| &= -|B| = -(-5) = 5 \end{aligned}$$

11 If we add the first row to the last row to simplify the determinant, we get

$$\begin{vmatrix} 3 & 1 & 5 \\ 9 & 3 & 15 \\ -3 & -1 & -5 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 5 \\ 9 & 3 & 15 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

12 We have that

$$A = \begin{pmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{pmatrix} \quad \Rightarrow \quad A^2 = \begin{pmatrix} b^2+c^2 & ab & ac \\ ab & a^2+c^2 & bc \\ ac & bc & a^2+b^2 \end{pmatrix}$$

This implies that

$$\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = |A|^2 = |A||A| = |AA| = |A^2| = \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & a^2+c^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix}$$

13 To determine which matrices are invertible, we calculate the determinants:

$$a) \quad \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0, \quad b) \quad \begin{vmatrix} 1 & 3 \\ -1 & 3 \end{vmatrix} = 6 \neq 0, \quad c) \quad \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

Hence the matrices in b) and c) are invertible, and we have

$$b) \quad \begin{pmatrix} 1 & 3 \\ -1 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{6} & \frac{1}{6} \end{pmatrix}, \quad c) \quad \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

14 In order to find the cofactor matrix, we must find all the cofactors of A:

$$C_{11} = (-1)^{1+1} \cdot \begin{vmatrix} 5 & 6 \\ 0 & 8 \end{vmatrix} = 40,\ C_{12} = (-1)^{1+2} \cdot \begin{vmatrix} 0 & 6 \\ 1 & 8 \end{vmatrix} = 6,\ C_{13} = (-1)^{3+1} \cdot \begin{vmatrix} 0 & 5 \\ 1 & 0 \end{vmatrix} = -5$$

$$C_{21} = (-1)^{2+1} \cdot \begin{vmatrix} 2 & 3 \\ 0 & 8 \end{vmatrix} = -16,\ C_{22} = (-1)^{2+2} \cdot \begin{vmatrix} 1 & 3 \\ 1 & 8 \end{vmatrix} = 5,\ C_{23} = (-1)^{2+3} \cdot \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 2$$

$$C_{31} = (-1)^{3+1} \cdot \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = -3,\ C_{32} = (-1)^{3+2} \cdot \begin{vmatrix} 1 & 3 \\ 0 & 6 \end{vmatrix} = -6,\ C_{33} = (-1)^{3+3} \cdot \begin{vmatrix} 1 & 2 \\ 0 & 5 \end{vmatrix} = 5$$

From this we find the cofactor matrix and the adjoint matrix of A:

$$\begin{pmatrix} 40 & 6 & -5 \\ -16 & 5 & 2 \\ -3 & -6 & 5 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 40 & 6 & -5 \\ -16 & 5 & 2 \\ -3 & -6 & 5 \end{pmatrix}^T = \begin{pmatrix} 40 & -16 & -3 \\ 6 & 5 & -6 \\ -5 & 2 & 5 \end{pmatrix}$$

The determinant $|A|$ of A is 37 from the problem above. The inverse matrix is then

$$A^{-1} = \frac{1}{37} \begin{pmatrix} 40 & -16 & -3 \\ 6 & 5 & -6 \\ -5 & 2 & 5 \end{pmatrix} = \begin{pmatrix} \frac{40}{37} & -\frac{16}{37} & -\frac{3}{37} \\ \frac{6}{37} & \frac{5}{37} & -\frac{6}{37} \\ -\frac{5}{37} & \frac{2}{37} & \frac{5}{37} \end{pmatrix}$$

Similarly, we find the cofactor matrix and the adjoint matrix of B to be

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b & 0 & 1 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b & 0 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & -b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We compute that $|B| = 1$, and it follows that B^{-1} is given by

$$B^{-1} = \begin{pmatrix} 1 & 0 & -b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We verify that $AA^{-1} = BB^{-1} = I$.

15 We note that

$$\begin{pmatrix} 5x_1 + x_2 \\ 2x_1 - x_2 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

This means that

$$\begin{aligned} 5x_1 + x_2 &= 3 \\ 2x_1 - x_2 &= 4 \end{aligned}$$

is equivalent to

$$\begin{pmatrix} 5 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

We thus have

$$A = \begin{pmatrix} 5 & 1 \\ 2 & -1 \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

Since $|A| = 5(-1) - 2 \cdot 1 = -7 \neq 0$, A is invertible. By the formula for the inverse of an 2×2-matrix, we get

$$A^{-1} = \begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{2}{7} & -\frac{5}{7} \end{pmatrix}.$$

If we multiply the matrix equation $A\mathbf{x} = \mathbf{b}$ on the left by A^{-1}, we obtain

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}.$$

Now, the important point is that $A^{-1}A = I$ and $I\mathbf{x} = \mathbf{x}$. Thus we get that $\mathbf{x} = A^{-1}\mathbf{b}$. From this we find the solution:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{2}{7} & -\frac{5}{7} \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

In other words $x_1 = 1$ and $x_2 = -2$.

16 (a) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$

17 Removing a column gives a 3-minor. Thus there are 4 minors of order 3. To get a 2-minor, we must remove a row and two columns. There are $3 \cdot 4 \cdot 3/2 = 18$ ways to do this, so there are 18 minors of order 2. The 1-minors are the entries of the matrix, so there are $3 \cdot 4 = 12$ minors of order 1.

18 (a) We compute the determinant

$$\begin{vmatrix} x & 0 & x^2-2 \\ 0 & 1 & 1 \\ -1 & x & x-1 \end{vmatrix} = x^2 - x - 2.$$

We have that $x^2 - x - 2 = 0$ if and only if $x = -1$ or $x = 2$, so if $x \neq -1$ and $x \neq 2$, then $r(A) = 3$. If $x = -1$, then

$$A = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & -1 & -2 \end{pmatrix}.$$

Since for instance $\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = -1 \neq 0$, it follows that $r(A) = 2$. If $x = 2$, then

$$A = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{pmatrix}.$$

Since for instance $\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = 2 \neq 0$, we see that $r(A) = 2$.

(b) We compute the determinant

$$\begin{vmatrix} t+3 & 5 & 6 \\ -1 & t-3 & -6 \\ 1 & 1 & t+4 \end{vmatrix} = (t+4)(t+2)(t-2)$$

Hence the rank is 3 if $t \neq -4$, $t \neq -2$, and $t \neq 2$. The rank is 2 if $t = -4$,$t = -2$, or $t = 2$, since there is a non-zero minor of order 2 in each case.

19 See answers to [FMEA] 1.3.3 on page 559.

20 See answers to [FMEA] 1.4.1 on page 559.

21 $A(\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) = \lambda A\mathbf{x}_1 + (1-\lambda)A\mathbf{x}_2 = \lambda\mathbf{b} + (1-\lambda)\mathbf{b} = \mathbf{b}$. This shows that if $\mathbf{x}_1$ and $\mathbf{x}_2$ are different solutions, then so are all points on the straight line through $\mathbf{x}_1$ and $\mathbf{x}_2$.

22 See answers to [FMEA] 1.4.6 on page 560.

23 Midterm Exam in GRA6035 24/09/2010, Problem 1
Since the augmented matrix of the system is in echelon form, we see that the system is consistent and has two free variables, x_3 and x_5. Hence the correct answer is alternative **4**.

24 Mock Midterm Exam in GRA6035 09/2010, Problem 1
Since the augmented matrix of the system is in echelon form, we see that the system is inconsistent. Hence the correct answer is alternative **4**.

25 Midterm Exam in GRA6035 24/05/2011, Problem 3
Since the augmented matrix of the system is in echelon form, we see that the system is inconsistent. Hence the correct answer is alternative **1**.