b) Lagrange Problem: $\boxed{\text{Revision Problems, Problem 2}}$

Problem 2: max $2z$ subject to $\begin{cases} x^2+y^2=2 \\ x+y+z=1 \end{cases}$

a) Lagrangian: $\mathcal{L} = 2z - \lambda_1(x^2+y^2) - \lambda_2(x+y+z)$

FOC: $$\boxed{\begin{aligned} \mathcal{L}'_x &= -\lambda_1 \cdot 2x - \lambda_2 = 0 \\ \mathcal{L}'_y &= -\lambda_1 \cdot 2y - \lambda_2 = 0 \\ \mathcal{L}'_z &= 2 - \lambda_2 = 0 \end{aligned}}$$

b) $\boxed{\text{FOC}}$ + $\boxed{\text{Constraints}}$:

$$\boxed{\begin{aligned} -\lambda_1 \cdot 2x - \lambda_2 &= 0 \\ -\lambda_1 \cdot 2y - \lambda_2 &= 0 \\ 2 - \lambda_2 &= 0 \\ x^2+y^2 &= 2 \\ x+y+z &= 1 \end{aligned}} \Rightarrow \lambda_2 = 2$$

$\Rightarrow -\lambda_1 \cdot 2x = 2 \Rightarrow x = -1/\lambda_1, \ \lambda_1 \neq 0$

$-\lambda_1 \cdot 2y = 2 \Rightarrow y = -1/\lambda_1, \ \lambda_1 \neq 0$

$\Downarrow$

$x = y$

$x^2+y^2=2 \Rightarrow x^2=y^2=1 \Rightarrow x=y=\pm 1$

$x=y=1 \Rightarrow \lambda_1 = -1 \Rightarrow z=-1$

$x=y=-1 \Rightarrow \lambda_1=+1 \Rightarrow z=3$

Solutions: $(x,y,z;\lambda_1,\lambda_2) = (1,1,-1;-1,2)$

$(-1,-1,3;\ 1,2)$

Values: $f(1,1,-1) = -2$

$f(-1,-1,3) = 6$

Candidates for max: $f=6$ at $(-1,-1,3;1,2)$

Theory: Convex/concave $\mathcal{L}$

Given $(x,y,z;\lambda_1,\lambda_2)$ that solves $\boxed{\text{FOC}}$ + $\boxed{\text{Constraints}}$. Consider

$\mathcal{L}(x,y,z)$ as a function of (x,y,z),
with λ_1 and λ_2 fixed from the points we look at

Then we have:

$$\boxed{\begin{aligned} &\mathcal{L} \text{ convex} \Rightarrow (x,y,z) \text{ is global min} \\ &\mathcal{L} \text{ concave} \Rightarrow (x,y,z) \text{ is global max} \end{aligned}}$$

(solves Lagrange Problem with min/max)

We use this theory at the candidate $(-1,-1,3;1,2)$:

$$\mathcal{L} = 2z - \lambda_1(x^2+y^2) - \lambda_2(x+y+z) = +2z - 1\cdot(x^2+y^2) - 2(x+y+z)$$
$$= \cancel{2z} - x^2 - y^2 - 2x - 2y - \cancel{2z} = \underline{-x^2-y^2-2x-2y}$$

This function is $\underline{\text{concave}}$ since $\mathcal{L}'' = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

This means that $\underline{(-1,-1,3) \text{ gives max}}$

$\underline{\text{max value } f=6}$

c) $\underline{\text{NDCQ}}$: $\boxed{\text{rk}\begin{pmatrix} 2x & 2y & 0 \\ 1 & 1 & 1 \end{pmatrix} = 2}$ ← the condition we should check

$\underline{\text{Check NDCQ}}$: $\text{rk}\begin{pmatrix} 2x & 2y & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{cases} 1, & x=y=0 \\ 2, & \text{otherwise} \end{cases}$

$x=y=0$ is not possible at admissible point, so $\text{rk}=2$
$\underline{\text{NDCQ satisfied for all admissible points}}$

For the solution method we used in b), it was $\underline{\text{not}}$ necessary to check NDCQ.

d) $\underline{x+y+z=b}$: $\boxed{\text{FOC}}$ + $\boxed{\text{Constraints}}$ give almost same solution as in b) (with $b=3$):

$\lambda_2 = 2 \Rightarrow x=y=-1/\lambda_1$

$x^2+y^2=2 \Rightarrow x=y=\pm 1$

$z = b-1-1 = b-2$ if $x=y=1$

$z = b+1+1 = b+2$ if $x=y=-1$

$\Downarrow$

$\underline{\text{Candidates}}$:

$(x,y,z;\lambda_1,\lambda_2) = (1,1,b-2;-1,2)$
$(-1,-1,b+2;1,2)$

$f = 2(b-2)$ in $(1,1,b-2)$
$f = 2(b+2)$ in $(-1,-1,b+2)$

$\underline{\text{Best candidate}}$: $\boxed{(-1,-1,b+2) \quad f=2b+4}$

In $(-1,-1,b+2;1,2)$, we still have

$\mathcal{L} = -x^2-y^2-2x-2y$ is concave

so $\underline{(x,y,z)=(-1,-1,b+2) \text{ is max}}$

$\underline{f=2b+4}$ is max value

If b increases, then max value

$f=2b+4$

increases $\left(\frac{\partial f}{\partial b} = 2 = \lambda_2\right)$.

c) Kuhn-Tucker Problem: $\boxed{\text{Revision Problems, Problem 3}}$

Problem 3: $\max 2z$ subject to $\begin{cases} x^2+y^2 \le 2 \\ x+y+z \le 1 \end{cases}$

Kuhn-Tucker problem in std form
(max problem $g_i(\underline{x}) \le b_i$)

Note: If the problem is not std form, change it before you start

min f subj. to ... → max −f subj. to ...
$g_i(\underline{x}) \ge b_i$ → $-g_i(\underline{x}) \le -b_i$

a) L and FOC the same as in Problem 2.
Complementary slackness conditions:

$$\boxed{\begin{array}{l} \lambda_1 \ge 0 \text{ and } (x^2+y^2<2 \Rightarrow \lambda_1 = 0) \\ \lambda_2 \ge 0 \text{ and } (x+y+z<1 \Rightarrow \lambda_2 = 0) \end{array}}$$

b) Solve $\boxed{\text{FOC}}$ + $\boxed{\text{Constraints}}$ + $\boxed{\text{Complementary Slackness Cond.}}$ = CSC

Divide in 4 cases:

$x^2+y^2=2$ $x+y+z=1$	$x^2+y^2=2$ $x+y+z<1$	$x^2+y^2<2$ $x+y+z=1$	$x^2+y^2<2$ $x+y+z<1$	Constraints
$-\lambda_1 \cdot 2x - \lambda_2 = 0$ $-\lambda_1 \cdot 2y - \lambda_2 = 0$ $2 - \lambda_2 = 0$	← same	← same	← same	FOC
$\lambda_1 \ge 0$ $\lambda_2 \ge 0$	$\lambda_1 \ge 0$ $\lambda_2 = 0$	$\lambda_1 = 0$ $\lambda_2 \ge 0$	$\lambda_1 = 0$ $\lambda_2 = 0$	CSC
From Problem 2: $(1,1,-1;-1,2)$ $(-1,-1,3;1,2)$ First is no soln. Since $\lambda_1 = -1 < 0$	no soln: $\lambda_2 = 2$ and $\lambda_2 = 0$	$\lambda_1 = 0$ $\lambda_2 = -\lambda_1 \cdot 2x = 0$ $\lambda_2 = 2 \Rightarrow$ impossible no soln	no soln: $\lambda_2 = 2$ and $\lambda_2 = 0$	

In conclusion, we find one candidate for max by solving

$$\boxed{\text{FOC}} + \boxed{\text{Constraints}} + \boxed{\text{CSC}}$$

The point $(x,y,z;\lambda_1,\lambda_2) = (-1,-1,3;1,2)$
with $f = 6$

Theory: Kuhn-Tucker problems
Solutions of $\boxed{\text{FOC}} + \boxed{\text{Constraints}} + \boxed{\text{CSC}}$ give candidates for max. We must do more to check that a point is actually max.

Method 1: Check that $L(x,y,z)$ is concave when we fix λ_1,λ_2 from a specific candidate point. In that case, the point is max.

Method 2: i) Make sure that there is a max. ← Extreme value theorem
ii) If so, max must be one of the following points

* Solutions of $\boxed{\text{FOC}} + \boxed{\text{Constraints}} + \boxed{\text{CSC}}$
* Admissible points that do no satisfy NDCQ

Make this list and compute f at each point.

The candidate $(-1,-1,3;1,2)$ gives concave L just as in Problem 2b).
Therefore $(-1,-1,3)$ is max
$f=6$ is max value.

c) NDCQ for Kuhn-Tucker problems: Check all the cases

$\left.\begin{matrix} x^2+y^2 \le 2 \\ x+y+z=1 \end{matrix}\right\}$ NDCQ: $\operatorname{rk}\begin{pmatrix} 2x & 2y & 0 \\ 1 & 1 & 1 \end{pmatrix} = 2$ ok for admissible points (same as Problem 2).

$\left.\begin{matrix} x^2+y^2 = 2 \\ x+y+z<1 \end{matrix}\right\}$ NDCQ: $\operatorname{rk}\begin{pmatrix} 2x & 2y & 0 \end{pmatrix} = 1$ ok for admissible points ($x=y=0$ not adm.)

$$\left.\begin{array}{l} x^2+y^2<2 \\ x+y+z=1 \end{array}\right\}$$ NDCQ: $\text{rk}\begin{pmatrix}1 & 1 & 1\end{pmatrix}=1$ ok.

$$\left.\begin{array}{l} x^2+y^2<2 \\ x+y+z<1 \end{array}\right\}$$ NDCQ: no condition to check ok.

Conclusion: NDCQ satisfied for all admissible points
Not necessary to check this to use Method I in b).

Theory: NDCQ for Kuhn-Tucker problems

Constraints:

$$\boxed{\begin{array}{l} g_1(\underline{x}) \le b_1 \\ g_2(\underline{x}) \le b_2 \\ \vdots \\ g_m(\underline{x}) \le b_m \end{array}}$$

Divide into all the different cases.
Include the row

$$\begin{pmatrix} \frac{\partial g_i}{\partial x_1} & \frac{\partial g_i}{\partial x_2} & \cdots & \frac{\partial g_i}{\partial x_n} \end{pmatrix}$$

in the matrix when $g_i(\underline{x}) = b_i$ (binding constraints)

NDCQ: rk (this matrix) = # rows (the matrix)
