Problem Sheet 10 with Solutions
GRA 6035 Mathematics

BI Norwegian Business School

Problems

1. Find the derivative $\dot{y}$ of the following functions:

a) $y = \frac{1}{2}t - \frac{3}{2}t^2 + 5t^3$
b) $y = (2t^2 - 1)(t^4 - 1)$
c) $y = (\ln t)^2 - 5\ln t + 6$
d) $y = \ln(3t)$
e) $y = 5e^{-3t^2+t}$
f) $y = 5t^2e^{-3t}$

2. Compute the following integrals:

a) $\int t^3 dt$
b) $\int_0^1 (t^3 + t^5 + \frac{1}{3})dt$
c) $\int \frac{1}{t} dt$
d) $\int te^{t^2} dt$
e) $\int \ln t dt$

3. Find the general solution, and the particular solution satisfying $y(0) = 1$ in the following differential equations:

a) $\dot{y} = 2t$.
b) $\dot{y} = e^{2t}$
c) $\dot{y} = (2t+1)e^{t^2+t}$
d) $\dot{y} = \frac{2t+1}{t^2+t+1}$

4. We consider the differential equation $\dot{y} + y = e^t$. Show that $y(t) = Ce^{-t} + \frac{1}{2}e^t$ is a solution of the differential equation for all values of the constant C.

5. Show that $y = Ct^2$ is a solution of $t\dot{y} = 2y$ for all choices of the constant C, and find the particular solution satisfying $y(1) = 2$.

6. Solve the differential equation $y^2\dot{y} = t + 1$, and find the integral curve that goes through the point $(t, y) = (1, 1)$.

7. Solve the following differential equations:

a) $\dot{y} = t^3 - 1$
b) $\dot{y} = te^t - t$
c) $e^y\dot{y} = t + 1$

8. Solve the following differential equations with initial conditions:

a) $t\dot{y} = y(1-t)$, with $(t_0, y_0) = (1, \frac{1}{e})$
b) $(1+t^3)\dot{y} = t^2y$, with $(t_0, y_0) = (0, 2)$
c) $y\dot{y} = t$, with $(t_0, y_0) = (\sqrt{2}, 1)$
d) $e^{2t}\dot{y} - y^2 - 2y = 1$, with $(t_0, y_0) = (0, 0)$

Challenging Optimization Problems for Advanced Students

These optimization problems are quite challenging and are meant for advanced students. It is recommended that you work through the ordinary problems and exam problems from Problem Sheet 5-9 and make sure that you master them before you attempt Problem 9 - 10 (which are optional).

9. Consider the following Kuhn-Tucker problem:

$$\max\ e^x(1+z) \quad \text{subject to} \quad \begin{cases} x^2+y^2 \le 1 \\ x+y+z \le 1 \end{cases}$$

Write down the Kuhn-Tucker conditions for this problem and solve them. Use the result to solve the optimization problem. (Hint: Even if the set of admissible points is not bounded, you could still find another argument to show that the problem must have a solution).

10. Consider the following Kuhn-Tucker problem:

$$\max\ 3xy - x^3 \quad \text{subject to} \quad \begin{cases} 2x - y \ge -5 \\ 5x + 2y \le 37 \\ x, y \ge 0 \end{cases}$$

a) Sketch the region in the xy-coordinate plane that satisfy all constraints, and use this to show that the region is bounded.
b) Write down the Kuhn-Tucker conditions, and solve the problem.
c) We replace the constraint $2x - y \ge -5$ with $2x - y = -5$, so that the optimization problem has *mixed constraints* — both equality and inequality constraints. Describe the changes we must make to the Kuhn-Tucker conditions to solve this new problem and explain why. Use this to solve the new problem.

Solutions

1

a) $\dot{y} = \frac{1}{2} - 3t + 15t^2$
b) $\dot{y} = 4t(t^4 - 1) + (2t^2 - 1)4t^3 = 12t^5 - 4t^3 - 4t$
c) $\dot{y} = 2(\ln t)\frac{1}{t} - 5\frac{1}{t}$
d) $\dot{y} = \frac{1}{t}$
e) $\dot{y} = 5e^{-3t^2+t}(-6t + 1)$
f) $\dot{y} = 10te^{-3t} - 15t^2e^{-3t}$

2

a) $\int t^3 dt = \frac{1}{4}t^4 + C$
b) $\int_0^1 (t^3 + t^5 + \frac{1}{3})dt = \frac{3}{4}$
c) $\int \frac{1}{t} dt = \ln|t| + C$
d) To find the integral $\int te^{t^2} dt$ we substitute $u = t^2$. This gives $\frac{du}{dt} = 2t$ or $\frac{du}{2} = tdt$. We get

$$\int te^{t^2} dt = \int e^u \frac{du}{2} = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C = \frac{1}{2}e^{t^2} + C$$

e) We use integration by parts

$$\int uv' dt = uv - \int u'v dt.$$

We write $\int \ln t dt$ as $\int (\ln t) \cdot 1 dt$ and let $u = \ln t$ and $v' = 1$. Thus $u' = \frac{1}{t}$ and $v = t$, and

$$\begin{aligned}\int \ln t dt &= (\ln t)t - \int \frac{1}{t} t dt \\ &= t \ln t - \int 1 dt \\ &= t \ln t - t + C\end{aligned}$$

3

a) $y = \int 2t dt = t^2 + C$. The general solution is $y = t^2 + C$. We get $y(0) = C = 1$, so $y = t^2 + 1$ is the particular solution satisfying $y(0) = 1$.
b) $y = \frac{1}{2}e^{2t} + C$ is the general solution. We get $y(0) = \frac{1}{2}e^{2\cdot 0} + C = \frac{1}{2} + C = 1 \implies C = \frac{1}{2}$. Thus $y(t) = \frac{1}{2}e^{2t} + \frac{1}{2}$ is the particular solution.
c) To find the integral $\int (2t+1)e^{t^2+t} dt$, we substitute $u = t^2 + t$. We get $\frac{du}{dt} = 2t + 1 \implies du = (2t+1)dt$, so

$$\int (2t+1)e^{t^2+t} dt = \int e^u du = e^u + C = e^{t^2+t} + C.$$

The general solution is $y = e^{t^2+t} + C$. This gives $y(0) = 1 + C = 1 \implies C = 0$. The particular solution is $y = e^{t^2+t}$.

d) We substitute $u = t^2 + t + 1$ in $\int \frac{2t+1}{t^2+t+1} dt$ to find the general solution $y = \ln(t^2 + t + 1) + C$. We get $y(0) = \ln 1 + C = C = 1$. The particular solution is $y(t) = \ln(t^2 + t + 1) + 1$.

4 $y(t) = Ce^{-t} + \frac{1}{2}e^t \implies \dot{y} = -Ce^{-t} + \frac{1}{2}e^t$. From this we get

$$\dot{y} + y = -Ce^{-t} + \frac{1}{2}e^t + Ce^{-t} + \frac{1}{2}e^t = e^t$$

so we see that $\dot{y} + y = e^t$ is satisfied when $y = Ce^{-t} + \frac{1}{2}e^t$.

5 $y = Ct^2 \implies \dot{y} = 2Ct$. We have

$$t\dot{y} = t \cdot 2Ct = 2Ct^2 = 2y$$

6 The equation $y^2\dot{y} = t + 1$ is separable:

$$y^2 \frac{dy}{dt} = t + 1$$

gives

$$\begin{aligned} \int y^2 dy &= \int (t+1) dt \\ \frac{1}{3}y^3 &= \frac{1}{2}t^2 + t + C \\ y^3 &= \frac{3}{2}t^2 + 3t + 3C \end{aligned}$$

Taking third root and renaming the constant

$$y(t) = \sqrt[3]{\frac{3}{2}t^2 + 3t + K}$$

We want the particular solution with $y(1) = 1$. We have

$$\begin{aligned} y(1) &= \sqrt[3]{\frac{3}{2}1^2 + 3 + K} \\ &= \sqrt[3]{K + \frac{9}{2}} = 1 \implies K + \frac{9}{2} = 1 \end{aligned}$$

We get $K = -\frac{7}{2}$. Thus

$$y(t) = \sqrt[3]{\frac{3}{2}t^2 + 3t - \frac{7}{2}}$$

is the particular solution.

7

a) $\dot{y} = t^3 - 1$ gives
$$y = \int (t^3 - 1)dt$$
We get
$$y = \frac{1}{4}t^4 - t + C.$$

b) We must evaluate the integral $\int (te^t - t)dt$. To evaluate $\int te^t dt$ we use integration by parts
$$\int uv' dt = uv - \int u'v dt.$$
with $v' = e^t$ and $u = t$. We get $u' = 1$ and $v = e^t$. Thus
$$\int te^t dt = te^t - \int e^t dt = te^t - e^t + C$$
We get
$$y = \int (te^t - t)dt = te^t - e^t - \frac{1}{2}t^2 + C$$

c) $e^y \dot{y} = t + 1$ is separated as
$$e^y dy = (t+1)dt \implies \int e^y dy = \int (t+1)dt$$
Thus we get
$$e^y = \frac{1}{2}t^2 + t + C.$$
Taking the natural logarithm on each side, we get
$$y(t) = \ln(\frac{1}{2}t^2 + t + C).$$

8

a) $t\dot{y} = y(1-t)$ is separated as
$$\frac{dy}{y} = \frac{1-t}{t}dt \implies \int \frac{dy}{y} = \int \frac{1-t}{t}dt$$
Note that $\frac{1-t}{t} = \frac{1}{t} - 1$, so
$$\ln|y| = \ln|t| - t + C$$
From this we get
$$e^{\ln|y|} = e^{\ln|t| - t + C} = e^{\ln|t|}e^{-t}e^C \implies |y| = |t|e^{-t}e^C$$
From this we deduce that

$$y(t) = te^{-t}K$$

where K is a constant as the general solution. We will find the particular solution with $y(1) = \frac{1}{e}$. We get

$$y(1) = e^{-1}K = e^{-1} \implies K = 1.$$

The particular solution is

$$y(t) = te^{-t}.$$

b) The equation $(1+t^3)\dot{y} = t^2 y$ is separated as

$$\frac{dy}{y} = \frac{t^2}{1+t^3}dt \implies \int \frac{dy}{y} = \int \frac{t^2}{1+t^3}dt$$

We get

$$\ln|y| = \frac{1}{3}\ln|1+t^3| + C = \ln|1+t^3|^{\frac{1}{3}} + C$$

This gives

$$e^{\ln|y|} = e^{\ln|1+t^3|^{\frac{1}{3}}+C}$$

This gives

$$|y| = |1+t^3|^{\frac{1}{3}}e^C$$

from which we deduce the general solution

$$y(t) = K(1+t^3)^{\frac{1}{3}}$$

where K is a constant. We which to find the particular solution with $y(0) = 2$. We get

$$y(0) = K = 2.$$

Thus the particular solution is

$$y(t) = 2(1+t^3)^{\frac{1}{3}}.$$

c) $y\dot{y} = t$ is separated as

$$ydy = tdt \implies \int ydy = \int tdt$$

The general solution is

$$y^2 = t^2 + C$$

where y is define implicitly. We want the particular solution where $y(\sqrt{2}) = 1$. We get

$$1^2 = (\sqrt{2})^2 + C \implies 1 = 2 + C \implies C = -1$$

We have

$$y^2 = t^2 - 1 \implies y = \pm\sqrt{t^2 - 1}$$

since $y(\sqrt{2}) < 0$ we have

$$y(t) = \sqrt{t^2 - 1}$$

as the particular solution.

d) $e^{2t}\frac{dy}{dt} - y^2 - 2y = 1$, is separated as follows:

$$e^{2t}\dot{y} - y^2 - 2y = 1 \implies e^{2t}\dot{y} = 1 + y^2 + 2y = (y+1)^2 \implies$$
$$\frac{dy}{(y+1)^2} = e^{-2t}dt \implies \int \frac{dy}{(y+1)^2} = \int e^{-2t}dt$$

To solve the integral

$$\int \frac{dy}{(y+1)^2}$$

we substitute $u = y + 1$. We get $\frac{du}{dy} = 1 \implies dy = du$. Thus

$$\int \frac{dy}{(y+1)^2} = \int \frac{1}{u^2}du = \int u^{-2}du = \frac{1}{-1}u^{-2+1} + C = -u^{-1} + C = -\frac{1}{(y+1)} + C$$

Thus we get

$$-\frac{1}{(y+1)} = \frac{1}{-2}e^{-2t} + C = -\frac{1}{2}e^{-2t} + C \implies -y - 1 = \frac{1}{-\frac{1}{2}e^{-2t} + C}$$

From this we get

$$y(t) = \frac{-1}{-\frac{1}{2}e^{-2t} + C} - 1$$

as the general solution. We want the particular solution with $y(0) = 0$. We get

$$y(0) = \frac{-1}{-\frac{1}{2}e^{0} + C} - 1 = 0$$

From this we get $C = -\frac{1}{2}$. Thus the particular solution is

$$\begin{aligned} y(t) &= \frac{-1}{-\frac{1}{2}e^{-2t} - \frac{1}{2}} - 1 \\ &= \frac{1 - e^{-2t}}{1 + e^{-2t}}. \end{aligned}$$