LECTURE 13 — GRA 6035

EIVIND ERIKSEN — NOV 22^{TH} 2012 — MATHEMATICS

Plan:

Reading: [FMEA] 11.3-11.4, 6.4

1. Difference equations
   * Revision: First order diff. eqn's
   * Second order difference equations
2. Stability
   * differential / difference equations
3. Revision:

   Overview over important methods and problems from the course ← Extra Lecture 1-6 (Lecture Notes from GRA 6035 2011)

Note: There will be an extra Problem Session

$\boxed{\text{Mon Dec } 10^{TH} \text{ at 14.00 - 17.00 in C1-000}}$

In this session, I will go through a Mock Exam (to be posted soon).

1 Difference equations

a) First order linear difference eqns:
(constant coeff.)

$$\boxed{y_{t+1} + a y_t = f_t}$$

a : constant

$f_t = 0$: homogeneous case

$f_t \neq 0$: general (inhomogeneous) case

Solution:

Homogeneous case

$$y_{t+1} + a y_t = 0 \Rightarrow y_t = \underline{C \cdot (-a)^t}$$
$$(y_{t+1} = -a \cdot y_t) \quad (\text{with } C = y_0)$$

Inhomogeneous case

$$y_{t+1} + a y_t = f_t \Rightarrow y_t = y_t^h + y_t^p = \underline{C \cdot (-a)^t + y_t^p}$$
$$(\text{with } C = y_0 - y_0^p)$$

b) Second order linear difference eqn's:
(constant coeff.)

$$\boxed{y_{t+2} + a y_{t+1} + b y_t = f_t}$$

a, b : constants

$f_t = 0$: homogeneous case

$f_t \neq 0$: inhomogeneous case

Homogeneous case:

$$y_{t+2} + a y_{t+1} + b y_t = 0$$

$y_t = r^t$ solution $\Leftrightarrow$ $r^2 + ar + b = 0$

Char. equation: $r^2 + ar + b = 0$

$$r = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

abc-formula

two, one or no real roots

$r_1 \neq r_2$ (two roots)	$y_t = C_1 r_1^t + C_2 r_2^t$	$\leftarrow a^2 - 4b > 0$
$r = -a/2$ (one root)	$y_t = C_1 r^t + C_2 t r^t = (C_1 + C_2 t) r^t$	$\leftarrow a^2 - 4b = 0$
(no roots)	$y_t = (\sqrt{b})^t \cdot (C_1 \cos \theta t + C_2 \sin \theta t)$ with $\theta = \cos^{-1}(-a/2\sqrt{b})$	$\leftarrow a^2 - 4b < 0$

Ex: $y_{t+2} = y_{t+1} + y_t \quad , y_0 = y_1 = 1$

Fibonacci sequence: $\underset{y_0}{1}, \underset{y_1}{1}, \underset{y_2}{2}, \underset{y_3}{3}, \underset{y_4}{5}, \underset{y_5}{8}, \underset{y_6}{13}, \underset{y_7}{21}, \underset{y_8}{34}, \ldots$

$$\boxed{y_{t+2} - y_{t+1} - y_t = 0}$$

Char. eqn: $r^2 - r - 1 = 0$

$$r = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$r_1 = \frac{1+\sqrt{5}}{2} \qquad r_2 = \frac{1-\sqrt{5}}{2}$$

General solution: $y_t = C_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^t + C_2 \cdot \left(\frac{1-\sqrt{5}}{2}\right)^t$

$y_0 = 1 \qquad C_1 \cdot 1 + C_2 \cdot 1 = 1 \qquad (t=0, y_t=1)$

$y_1 = 1 \qquad C_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right) + C_2 \cdot \left(\frac{1-\sqrt{5}}{2}\right) = 1 \quad \Big| \cdot 2 \qquad (t=1, y_t=1)$

$$C_1 + C_2 = 1 \Rightarrow C_2 = 1 - C_1$$

$$C_1(1+\sqrt{5}) + C_2 \cdot (1-\sqrt{5}) = 2$$

$$C_1(1+\sqrt{5}) + (1-C_1)(1-\sqrt{5}) = 2$$

$$C_1(\cancel{1} + \sqrt{5} \cancel{-1} + \sqrt{5}) = 2 - 1 + \sqrt{5}$$

$$2\sqrt{5}\, C_1 = 1 + \sqrt{5}$$

$$C_1 = \frac{1+\sqrt{5}}{2\sqrt{5}}$$

$$C_2 = 1 - \frac{1+\sqrt{5}}{2\sqrt{5}} = \frac{2\sqrt{5} - 1 - \sqrt{5}}{2\sqrt{5}}$$

$$= \frac{\sqrt{5}-1}{2\sqrt{5}}$$

$$y_t = \frac{\sqrt{5}+1}{2\sqrt{5}} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^t + \frac{\sqrt{5}-1}{2\sqrt{5}} \cdot \left(\frac{1-\sqrt{5}}{2}\right)^t, \quad t = 0, 1, 2, \ldots$$

Inhomogeneous case

$$y_{t+2} + a y_{t+1} + b y_t = f_t \qquad (f_t \neq 0)$$

Superposition principle:

The general solution is : $y_t = y_t^h + y_t^p$

general solution of homogeneous eqn.

$y_{t+2} + a y_{t+1} + b y_t = 0$

particular solution of

$y_{t+2} + a y_{t+1} + b y_t = f_t$

Ex: $y_{t+2} - y_{t+1} - 2y_t = 3^t$

General solution: $y_t = y_t^h + y_t^p = C_1 \cdot 2^t + C_2 (-1)^t + \frac{1}{4} \cdot 3^t$

y_t^h:

$$y_{t+2} - y_{t+1} - 2y_t = 0$$

$$r^2 - r - 2 = 0 \Rightarrow r = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} = 2, -1$$

$$y_t^h = C_1 \cdot 2^t + C_2 \cdot (-1)^t$$

y_t^p:

$$y_{t+2} - y_{t+1} - 2y_t = 3^t$$

$$A \cdot 3^{t+2} - A \cdot 3^{t+1} - 2A \cdot 3^t = 3^t$$

$$3^t \cdot 9A - 3^t \cdot 3A - 3^t \cdot 2A = 3^t$$

$$\cancel{3^t} \cdot (9A - 3A - 2A) = \cancel{3^t}$$

$$4A = 1 \Rightarrow A = 1/4$$

$f_t = 3^t$

$f_{t+1} = 3^{t+1} = 3^t \cdot 3$

$f_{t+2} = 3^{t+2} = 3^t \cdot 9$

Guess: $y_t = A \cdot 3^t$

$$y_t^p = \frac{1}{4} \cdot 3^t$$

Method of undetermined coefficients: How to guess y_t^p

Starting point: f_t

* Look at f_t, f_{t+1}, f_{t+2} and find y_t^p which has the same form as f_t, f_{t+1}, f_{t+2} and contains undetermined coeffs.

* Verify if the guess fits in the difference eqn. for any values of the undetermined coeff.'s.

* If there are no solutions, try to multiply your guess with t.

Ex: $y_{t+2} - 7y_{t+1} + 12y_t = 3^t$

$$y_t = y_t^h + y_t^p = C_1\cdot 3^t + C_2\cdot 4^t - \frac{1}{3}t\cdot 3^t$$

y_t^h: $r^2 - 7r + 12 = 0$

$r = 3,\ r = 4 \Rightarrow y_t^h = C_1\cdot 3^t + C_2\cdot 4^t$

y_t^p: $y_t = A\cdot 3^t \Rightarrow A\cdot 3^{t+2} - 7\cdot A\cdot 3^{t+1} + 12A\,3^t = 3^t$

$(9A - 21A + 12A)\cdot 3^t = 3^t$

$(0A)\cdot 3^t = 3^t$

$0\cdot A = 1$

no solution

Try to multiply the guess with t:

$y_t = A\cdot t\cdot 3^t \Rightarrow A\cdot(t+2)\cdot 3^{t+2} - 7\cdot A(t+1)\cdot 3^{t+1} + 12A\,t\cdot 3^t = 3^t$

$\quad 18A \qquad -21A$

$\cancel{3^t}\cdot\big(9A(t+2) - 7A\cdot 3\cdot(t+1) + 12A\cdot t\big) = \cancel{3^t}$

$\cancel{(0A)\cdot t} + (-3A) = 1 \Rightarrow A = -1/3$

[FMEA] 6.4, 11.4

② Stability   Difference / differential equations

a) $y'' + ay' + by = f(t)$

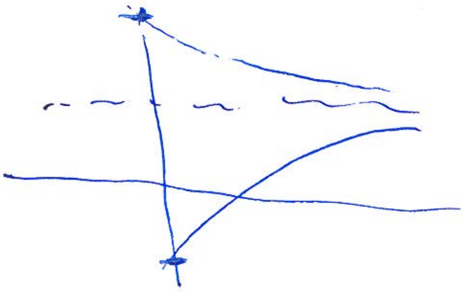
(second order linear differential eqn.)

$y(t) = y_h(t) + y_p(t) = \underbrace{C_1 \cdot u_1(t) + C_2 u_2(t)}_{y_h(t)} + y_p(t)$

initial conditions

$\Downarrow$

determine C_1, C_2



system globally asymptotically stable if the following condition holds:

$$\lim_{t\to\infty} y_h(t) = \lim_{t\to\infty} C_1 u_1(t) + C_2 u_2(t) = 0$$

the system is globally asymptotically stable if and only if

① $a^2 - 4b > 0$: $r_1, r_2 < 0$

② $a^2 - 4b = 0$: $r = -\frac{a}{2} < 0$

③ $a^2 - 4b < 0$: $r = -\frac{a}{2} \pm \frac{\sqrt{a^2-4b}}{2}$

condition: $-\frac{a}{2} < 0$

That is, the real parts of the characteric roots are negative.

Ex: $y'' - 7y' + 12y = 4$

$y = \underbrace{C_1 e^{3t} + C_2 e^{4t}} + \frac{1}{3}$

not gl. asympt. stable

$(r_1 = 3, r_2 = 4 > 0)$

b) $y_{t+2} + ay_{t+1} + by_t = f_t$

$y_t = y_t^h + y_t^p = \underbrace{C_1 \cdot u_t^1 + C_2 \cdot u_t^2} + y_t^p$

system globally asymptotically stable if and only if the following condition holds:

$$\lim_{t\to\infty} C_1 u_t^1 + C_2 u_t^2 = 0$$

Result:

system is gl. asymp. stable if and only if

$|r| < 1$

for each characteristic root

($r = \sqrt{b}$ in the case of no real char. roots)

Revision:

A) Matrix methods

B) Optimization – basics

C) Differential and difference equations

D) Optimization – advanced

A) Matrix methods

i) Compute determinants (and minors)
ii) Compute ranks (square and non-square case)
iii) Linear independence of vectors (and link with rank)
iv) Finding eigenvalues and eigenvectors
v) Diagonalization of matrices
vi) Determine definiteness of symmetric matrices

B) Optimization – basics

i) Compute partial derivatives and Hessian matrices (also for exp/ln)
ii) Determine if a function is convex/concave
iii) Find stationary pts and classify them (as local min/max/saddle pts)
iv) Know the Extreme Value Theorem and how to use it (also to determine if a set is bounded)
v) Know the Lagrange and Kuhn-Tucker conditions, and to be able to find solutions in basic problems
vi) Envelope theorems (unconstrained and constrained case)

C) Differential and difference equations

i) Integration (incl. integration by parts, substitution)

ii) Separable diff. eqn. ($y' = f(y) \cdot g(t)$)

iii) Linear first order diff. eqn. (integrating factor) ($y' + a(t) \cdot y = b(t)$)

iv) Exact first order diff. eqn.

v) First and second order linear homogeneous (char. equation)

vi) Linear inhomogeneous diff. eqn. (undetermined coeff.)

vii) First and second order linear homogeneous difference eqn. (char. equation)

viii) Linear inhomogeneous difference eqn. (undetermined coeff.)

D) Optimization - advanced

i) Methods to verify that solutions to Lagrange / Kuhn-Tucker conditions are optimal (convexity/concavity or extreme value thm + NDCQ)

ii) Classification of solutions to Lagrange ~~problems~~ conditions as local max/min (Bordered Hessian)

iii) Find solutions to Lagrange / Kuhn-Tucker conditions in difficult cases

iv) Solve optimization problems when standard methods do not work.

Some common misconceptions

(A) To figure out if a function $f(x_1, \ldots x_n)$ is convex or concave, when f is defined in D_f:

f convex $\Leftrightarrow$ $f''(x_1, \ldots, x_n)$ positive semidefinite for all $(x_1, \ldots, x_n) \in D_f$
f concave $\Leftrightarrow$ $f''(x_1, \ldots, x_n)$ negative semidefinite for all $(x_1, \ldots, x_n) \in D_f$

(B) To figure out if a stationary point $(x_1^*, \ldots, x_n^*)$ for f is local min/max/saddle pt:

$f''(x_1^*, \ldots, x_n^*)$ positive definite $\Rightarrow (x_1^*, \ldots, x_n^*)$ local min
$f''(x_1^*, \ldots, x_n^*)$ negative definite $\Rightarrow (x_1^*, \ldots, x_n^*)$ local max
— " — indefinite $\Rightarrow (x_1^*, \ldots, x_n^*)$ saddle point

Ex A: $f = x^2 + y^2 - 2x^3$ not convex/concave since $f''(x,y) = \begin{pmatrix} 2-12x^2 & 0 \\ 0 & 2 \end{pmatrix}$

and $D_1 = 2 - 12x^2$ can be both positive and negative

Ex B: $f = x^3 + y^3$ has stationary pt. at $(x,y) = (0,0)$ since $\begin{bmatrix} f'_x = 3x^2 = 0 \\ f'_y = 3y^2 = 0 \end{bmatrix}$ has solution $(x,y) = (0,0)$. The Hessian at $(0,0)$

is
$$f''(0,0) = \begin{pmatrix} 6x & 0 \\ 0 & 6y \end{pmatrix}\Bigg|_{(x,y)=(0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

So $f''(0,0)$ is both pos. and neg. semidefinite. Cannot use this test to determine if $(0,0)$ is local min/max/saddle pt.

(It is saddle pt, since $\begin{cases} f(0,0) = 0 \\ f(\varepsilon, 0) = \varepsilon^3 > 0 & \text{when } \varepsilon > 0 \text{ is small} \\ f(-\varepsilon, 0) = -\varepsilon^3 < 0 & \text{— " —} \end{cases}$)